Investigating Geometry **ACTIVITY** *Use before Lesson 5.4*

5.4 Intersecting Medians

MATERIALS • cardboard • straightedge • scissors • metric ruler

QUESTION What is the relationship between segments formed by the medians of a triangle?

EXPLORE 1 Find the balance point of a triangle

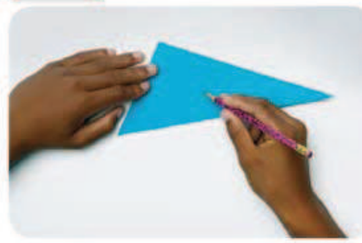
STEP 1

Cut out triangle Draw a triangle on a piece of cardboard. Then cut it out.

STEP 2

Balance the triangle Balance the triangle on the eraser end of a pencil.

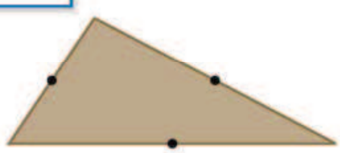
STEP 3



Mark the balance point Mark the point on the triangle where it balanced on the pencil.

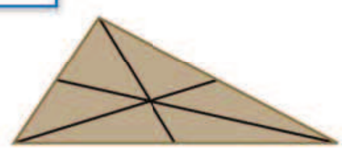
EXPLORE 2 Construct the medians of a triangle

STEP 1



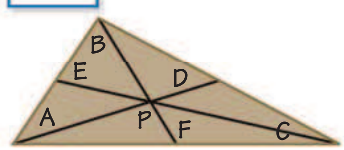
Find the midpoint Use a ruler to find the midpoint of each side of the triangle.

STEP 2



Draw medians Draw a segment, or *median*, from each midpoint to the vertex of the opposite angle.

STEP 3



Label points Label your triangle as shown. What do you notice about point *P* and the balance point in Explore 1?

DRAW CONCLUSIONS Use your observations to complete these exercises

1. Copy and complete the table. Measure in millimeters.

Length of segment from vertex to midpoint of opposite side	$AD = ?$	$BF = ?$	$CE = ?$
Length of segment from vertex to *P*	$AP = ?$	$BP = ?$	$CP = ?$
Length of segment from *P* to midpoint	$PD = ?$	$PF = ?$	$PE = ?$

2. How does the length of the segment from a vertex to *P* compare with the length of the segment from *P* to the midpoint of the opposite side?

3. How does the length of the segment from a vertex to *P* compare with the length of the segment from the vertex to the midpoint of the opposite side?

5.4 Use Medians and Altitudes

Before You used perpendicular bisectors and angle bisectors of triangles.

Now You will use medians and altitudes of triangles.

Why? So you can find the balancing point of a triangle, as in Ex. 37.

Key Vocabulary
- **median of a triangle**
- **centroid**
- **altitude of a triangle**
- **orthocenter**

As shown by the Activity on page 318, a triangle will balance at a particular point. This point is the intersection of the *medians* of the triangle.

A **median of a triangle** is a segment from a vertex to the midpoint of the opposite side. The three medians of a triangle are concurrent. The point of concurrency, called the **centroid**, is inside the triangle.

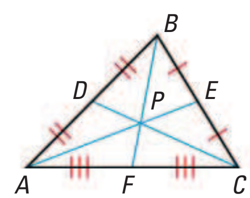
Three medians meet at the centroid.

THEOREM *For Your Notebook*

THEOREM 5.8 Concurrency of Medians of a Triangle

The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.

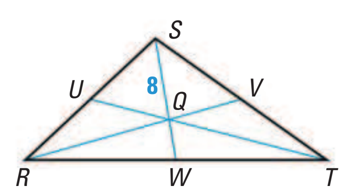
The medians of $\triangle ABC$ meet at P and $AP = \frac{2}{3}AE$, $BP = \frac{2}{3}BF$, and $CP = \frac{2}{3}CD$.



Proof: Ex. 32, p. 323; p. 934

EXAMPLE 1 Use the centroid of a triangle

In $\triangle RST$, Q is the centroid and $SQ = 8$. Find QW and SW.



Solution

$SQ = \frac{2}{3}SW$ — Concurrency of Medians of a Triangle Theorem

$8 = \frac{2}{3}SW$ — Substitute 8 for SQ.

$12 = SW$ — Multiply each side by the reciprocal, $\frac{3}{2}$.

Then $QW = SW - SQ = 12 - 8 = 4$.

▸ So, $QW = 4$ and $SW = 12$.

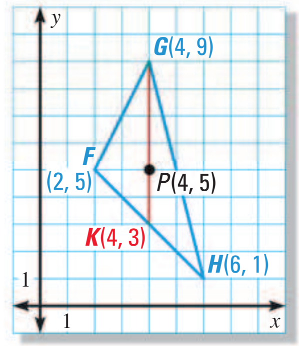
EXAMPLE 2 Standardized Test Practice

The vertices of $\triangle FGH$ are $F(2, 5)$, $G(4, 9)$, and $H(6, 1)$. Which ordered pair gives the coordinates of the centroid P of $\triangle FGH$?

Ⓐ (3, 5) Ⓑ (4, 5) Ⓒ (4, 7) Ⓓ (5, 3)

Solution

Sketch $\triangle FGH$. Then use the Midpoint Formula to find the midpoint K of $\overline{FH}$ and sketch median $\overline{GK}$.



$$K\left(\frac{2+6}{2}, \frac{5+1}{2}\right) = K(4, 3).$$

The centroid is two thirds of the distance from each vertex to the midpoint of the opposite side.

The distance from vertex $G(4, 9)$ to $K(4, 3)$ is $9 - 3 = 6$ units. So, the centroid is $\frac{2}{3}(6) = 4$ units down from G on $\overline{GK}$.

The coordinates of the centroid P are $(4, 9 - 4)$, or $(4, 5)$.

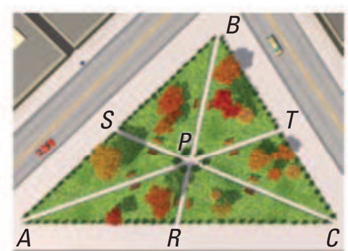
▶ The correct answer is B. Ⓐ Ⓑ Ⓒ Ⓓ

CHECK ANSWERS

Median $\overline{GK}$ was used in Example 2 because it is easy to find distances on a vertical segment. It is a good idea to check by finding the centroid using a different median.

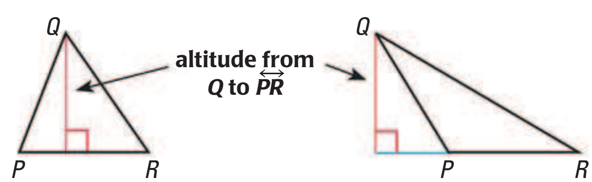
✓ GUIDED PRACTICE for Examples 1 and 2

There are three paths through a triangular park. Each path goes from the midpoint of one edge to the opposite corner. The paths meet at point P.



1. If $SC = 2100$ feet, find PS and PC.
2. If $BT = 1000$ feet, find TC and BC.
3. If $PT = 800$ feet, find PA and TA.

ALTITUDES An **altitude of a triangle** is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side.



MEASURES OF TRIANGLES

In the area formula for a triangle, $A = \frac{1}{2}bh$, you can use the length of any side for the base b. The height h is the length of the altitude to that side from the opposite vertex.

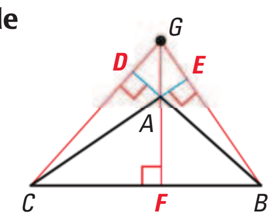
THEOREM *For Your Notebook*

THEOREM 5.9 Concurrency of Altitudes of a Triangle

The lines containing the altitudes of a triangle are concurrent.

The lines containing $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ meet at G.

Proof: Exs. 29–31, p. 323; p. 936



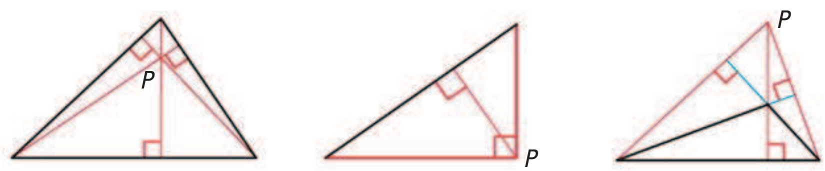
CONCURRENCY OF ALTITUDES The point at which the lines containing the three altitudes of a triangle intersect is called the **orthocenter** of the triangle.

EXAMPLE 3 Find the orthocenter

Find the orthocenter P in an acute, a right, and an obtuse triangle.

READ DIAGRAMS
The altitudes are shown in red. Notice that in the right triangle the legs are also altitudes. The altitudes of the obtuse triangle are extended to find the orthocenter.

Solution



Acute triangle
***P* is inside triangle.**

Right triangle
***P* is on triangle.**

Obtuse triangle
***P* is outside triangle.**

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ISOSCELES TRIANGLES In an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude from the vertex angle to the base are all the same segment. In an equilateral triangle, this is true for the special segment from any vertex.

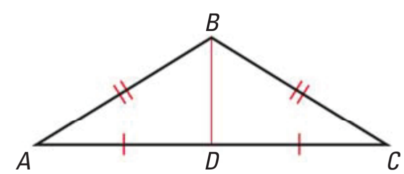
EXAMPLE 4 Prove a property of isosceles triangles

Prove that the median to the base of an isosceles triangle is an altitude.

Solution

GIVEN ▶ $\triangle ABC$ is isosceles, with base $\overline{AC}$.
$\overline{BD}$ is the median to base $\overline{AC}$.

PROVE ▶ $\overline{BD}$ is an altitude of $\triangle ABC$.



Proof Legs $\overline{AB}$ and $\overline{BC}$ of isosceles $\triangle ABC$ are congruent. $\overline{CD} \cong \overline{AD}$ because $\overline{BD}$ is the median to $\overline{AC}$. Also, $\overline{BD} \cong \overline{BD}$. Therefore, $\triangle ABD \cong \triangle CBD$ by the SSS Congruence Postulate.

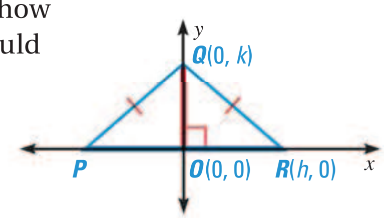
$\angle ADB \cong \angle CDB$ because corresponding parts of $\cong$ ⩓ are $\cong$. Also, $\angle ADB$ and $\angle CDB$ are a linear pair. $\overline{BD}$ and $\overline{AC}$ intersect to form a linear pair of congruent angles, so $\overline{BD} \perp \overline{AC}$ and $\overline{BD}$ is an altitude of $\triangle ABC$.

✓ GUIDED PRACTICE for Examples 3 and 4

4. Copy the triangle in Example 4 and find its orthocenter.

5. **WHAT IF?** In Example 4, suppose you wanted to show that median $\overline{BD}$ is also an angle bisector. How would your proof be different?

6. Triangle PQR is an isosceles triangle and segment $\overline{OQ}$ is an altitude. What else do you know about $\overline{OQ}$? What are the coordinates of P?



5.4 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 5, 21, and 39

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 7, 11, 12, 28, 40, and 44

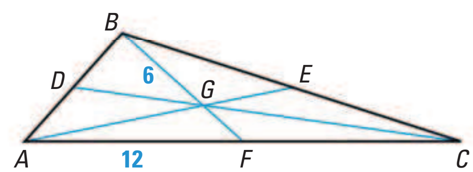
SKILL PRACTICE

1. **VOCABULARY** Name the four types of points of concurrency introduced in Lessons 5.2–5.4. When is each type inside the triangle? on the triangle? outside the triangle?

2. ★ **WRITING** *Compare* a perpendicular bisector and an altitude of a triangle. *Compare* a perpendicular bisector and a median of a triangle.

EXAMPLE 1 on p. 319 for Exs. 3–7

FINDING LENGTHS ***G* is the centroid of $\triangle ABC$, $BG = 6$, $AF = 12$, and $AE = 15$. Find the length of the segment.**



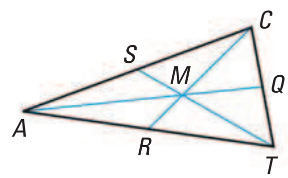
3. $\overline{FC}$

4. $\overline{BF}$

5. $\overline{AG}$

6. $\overline{GE}$

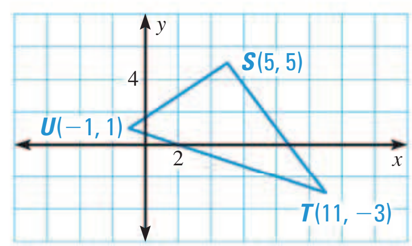
7. ★ **MULTIPLE CHOICE** In the diagram, M is the centroid of $\triangle ACT$, $CM = 36$, $MQ = 30$, and $TS = 56$. What is AM?



Ⓐ 15 Ⓑ 30 Ⓒ 36 Ⓓ 60

EXAMPLE 2 on p. 320 for Exs. 8–11

8. **FINDING A CENTROID** Use the graph shown.



   a. Find the coordinates of P, the midpoint of $\overline{ST}$. Use the median $\overline{UP}$ to find the coordinates of the centroid Q.

   b. Find the coordinates of R, the midpoint of $\overline{TU}$. Verify that $SQ = \frac{2}{3}SR$.

GRAPHING CENTROIDS **Find the coordinates of the centroid *P* of $\triangle ABC$.**

9. $A(-1, 2)$, $B(5, 6)$, $C(5, -2)$

10. $A(0, 4)$, $B(3, 10)$, $C(6, -2)$

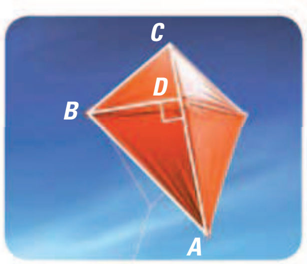
11. ★ **OPEN-ENDED MATH** Draw a large right triangle and find its centroid.

EXAMPLE 3 on p. 321 for Exs. 12–16

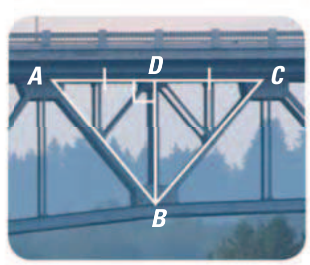
12. ★ **OPEN-ENDED MATH** Draw a large obtuse, scalene triangle and find its orthocenter.

IDENTIFYING SEGMENTS **Is $\overline{BD}$ a *perpendicular bisector* of $\triangle ABC$? Is $\overline{BD}$ a *median*? an *altitude*?**

13.

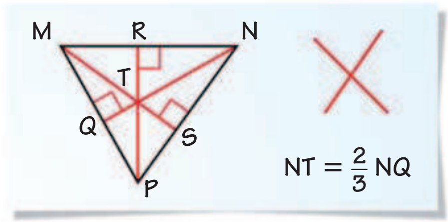


14.



15.

16. **ERROR ANALYSIS** A student uses the fact that T is a point of concurrency to conclude that $NT = \frac{2}{3}NQ$. *Explain* what is wrong with this reasoning.



EXAMPLE 4
on p. 321
for Exs. 17–22

REASONING **Use the diagram shown and the given information to decide whether $\overline{YW}$ is a *perpendicular bisector*, an *angle bisector*, a *median*, or an *altitude* of $\triangle XYZ$. There may be more than one right answer.**

17. $\overline{YW} \perp \overline{XZ}$

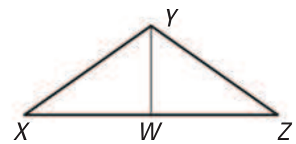
18. $\angle XYW \cong \angle ZYW$

19. $\overline{XW} \cong \overline{ZW}$

20. $\overline{YW} \perp \overline{XZ}$ and $\overline{XW} \cong \overline{ZW}$

21. $\triangle XYW \cong \triangle ZYW$

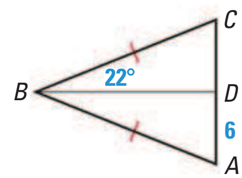
22. $\overline{YW} \perp \overline{XZ}$ and $\overline{XY} \cong \overline{ZY}$



ISOSCELES TRIANGLES **Find the measurements. *Explain* your reasoning.**

23. Given that $\overline{DB} \perp \overline{AC}$, find DC and $m\angle ABD$.

24. Given that $AD = DC$, find $m\angle ADB$ and $m\angle ABD$.



RELATING LENGTHS **Copy and complete the statement for $\triangle DEF$ with medians $\overline{DH}$, $\overline{EJ}$, and $\overline{FG}$, and centroid K.**

25. $EJ = \underline{\ ?\ }\ KJ$

26. $DK = \underline{\ ?\ }\ KH$

27. $FG = \underline{\ ?\ }\ KF$

28. ★ **SHORT RESPONSE** Any isosceles triangle can be placed in the coordinate plane with its base on the x-axis and the opposite vertex on the y-axis as in Guided Practice Exercise 6 on page 321. *Explain* why.

CONSTRUCTION **Verify the Concurrency of Altitudes of a Triangle by drawing a triangle of the given type and constructing its altitudes.** (*Hint:* To construct an altitude, use the construction in Exercise 25 on page 195.)

29. Equilateral triangle

30. Right scalene triangle

31. Obtuse isosceles triangle

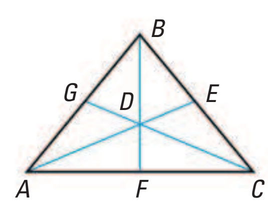
32. **VERIFYING THEOREM 5.8** Use Example 2 on page 320. Verify that Theorem 5.8, the Concurrency of Medians of a Triangle, holds for the median from vertex F and for the median from vertex H.

xy **ALGEBRA** **Point D is the centroid of $\triangle ABC$. Use the given information to find the value of x.**

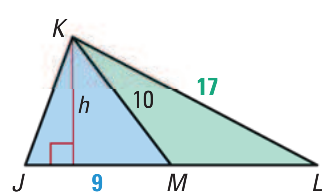
33. $BD = 4x + 5$ and $BF = 9x$

34. $GD = 2x - 8$ and $GC = 3x + 3$

35. $AD = 5x$ and $DE = 3x - 2$

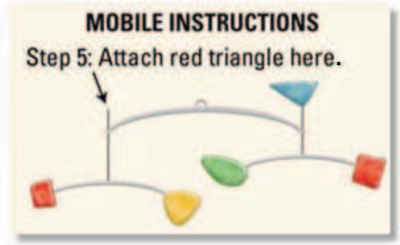


36. **CHALLENGE** $\overline{KM}$ is a median of $\triangle JKL$. Find the areas of $\triangle JKM$ and $\triangle LKM$. Compare the areas. Do you think that the two areas will always compare in this way, regardless of the shape of the triangle? *Explain*.



PROBLEM SOLVING

37. **MOBILES** To complete the mobile, you need to balance the red triangle on the tip of a metal rod. Copy the triangle and decide if you should place the rod at A or B. *Explain.*



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38. **DEVELOPING PROOF** Show two different ways that you can place an isosceles triangle with base $2n$ and height h on the coordinate plane. Label the coordinates for each vertex.

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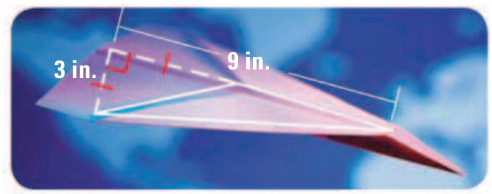
39. **PAPER AIRPLANE** Find the area of the triangular part of the paper airplane wing that is outlined in red. Which special segment of the triangle did you use?



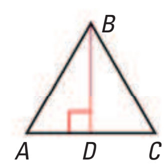
40. ★ **SHORT RESPONSE** In what type(s) of triangle can a vertex of the triangle be one of the points of concurrency of the triangle? *Explain.*

41. **COORDINATE GEOMETRY** Graph the lines on the same coordinate plane and find the centroid of the triangle formed by their intersections.

$y_1 = 3x - 4$ $\qquad$ $y_2 = \frac{3}{4}x + 5$ $\qquad$ $y_3 = -\frac{3}{2}x - 4$

EXAMPLE 4 on p. 321 for Ex. 42

42. **PROOF** Write proofs using different methods.



GIVEN ▶ $\triangle ABC$ is equilateral.
$\overline{BD}$ is an altitude of $\triangle ABC$.

PROVE ▶ $\overline{BD}$ is also a perpendicular bisector of $\overline{AC}$.

a. Write a proof using congruent triangles.

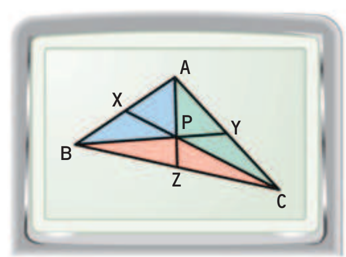
b. Write a proof using the Perpendicular Postulate on page 148.

43. **TECHNOLOGY** Use geometry drawing software.

a. Construct a triangle and its medians. Measure the areas of the blue, green, and red triangles.

b. What do you notice about the triangles?

c. If a triangle is of uniform thickness, what can you conclude about the weight of the three interior triangles? How does this support the idea that a triangle will balance on its centroid?



44. ★ **EXTENDED RESPONSE** Use $P(0, 0)$, $Q(8, 12)$, and $R(14, 0)$.

a. What is the slope of the altitude from R to $\overline{PQ}$?

b. Write an equation for each altitude of $\triangle PQR$. Find the orthocenter by finding the ordered pair that is a solution of the three equations.

c. How would your steps change if you were finding the circumcenter?

○ = WORKED-OUT SOLUTIONS on p. WS1 $\qquad$ ★ = STANDARDIZED TEST PRACTICE

45. **CHALLENGE** Prove the results in parts (a) – (c).

GIVEN ▶ $\overline{LP}$ and $\overline{MQ}$ are medians of scalene $\triangle LMN$. Point R is on $\overrightarrow{LP}$ such that $\overline{LP} \cong \overline{PR}$. Point S is on $\overrightarrow{MQ}$ such that $\overline{MQ} \cong \overline{QS}$.

PROVE ▶ **a.** $\overline{NS} \cong \overline{NR}$

b. $\overline{NS}$ and $\overline{NR}$ are both parallel to $\overline{LM}$.

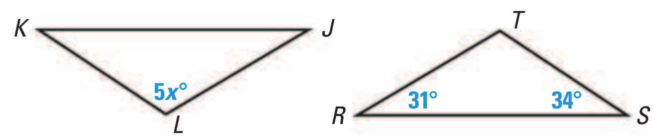
c. R, N, and S are collinear.

MIXED REVIEW

In Exercises 46–48, write an equation of the line that passes through points A and B. *(p. 180)*

46. $A(0, 7), B(1, 10)$ **47.** $A(4, -8), B(-2, -5)$ **48.** $A(5, -21), B(0, 4)$

49. In the diagram, $\triangle JKL \cong \triangle RST$. Find the value of x. *(p. 225)*



PREVIEW
Prepare for Lesson 5.5 in Exs. 50–52.

Solve the inequality. *(p. 287)*

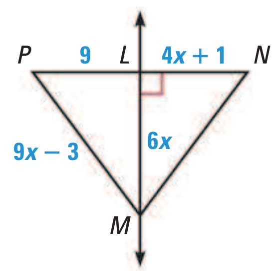
50. $2x + 13 < 35$ **51.** $12 > -3x - 6$ **52.** $6x < x + 20$

In the diagram, $\overline{LM}$ is the perpendicular bisector of $\overline{PN}$. *(p. 303)*

53. What segment lengths are equal?

54. What is the value of x?

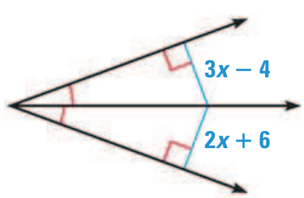
55. Find MN.



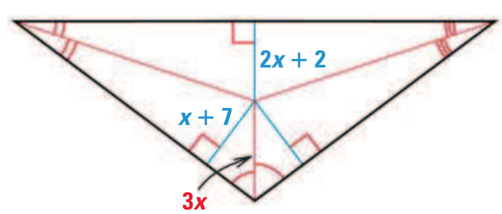
QUIZ for Lessons 5.3–5.4

Find the value of x. Identify the theorem used to find the answer. *(p. 310)*

1.



2.

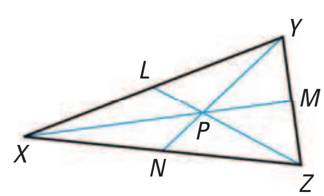


In the figure, P is the centroid of $\triangle XYZ$, $YP = 12$, $LX = 15$, and $LZ = 18$. *(p. 319)*

3. Find the length of $\overline{LY}$.

4. Find the length of $\overline{YN}$.

5. Find the length of $\overline{LP}$.



Technology **ACTIVITY** *Use after Lesson 5.4*

5.4 Investigate Points of Concurrency

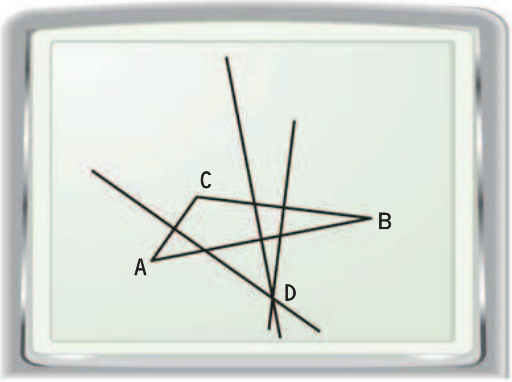
MATERIALS • graphing calculator or computer

QUESTION How are the points of concurrency in a triangle related?

You can use geometry drawing software to investigate concurrency.

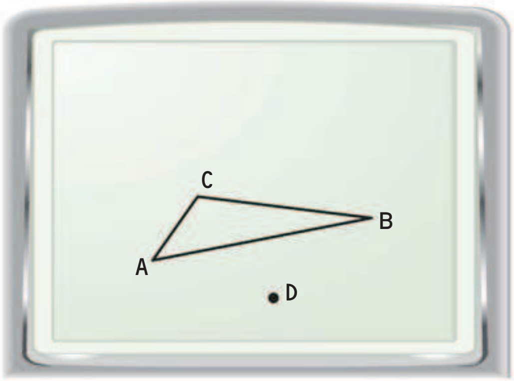
EXAMPLE 1 Draw the perpendicular bisectors of a triangle

STEP 1



Draw perpendicular bisectors Draw a line perpendicular to each side of a $\triangle ABC$ at the midpoint. Label the point of concurrency D.

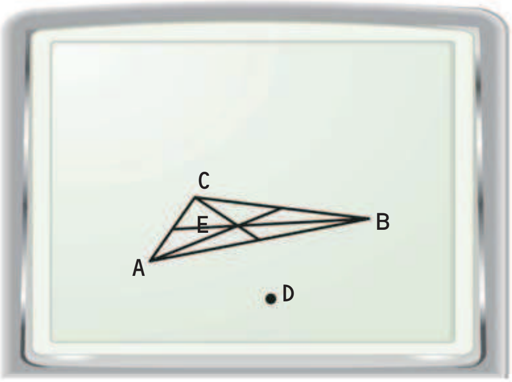
STEP 2



Hide the lines Use the *HIDE* feature to hide the perpendicular bisectors. Save as "EXAMPLE1."

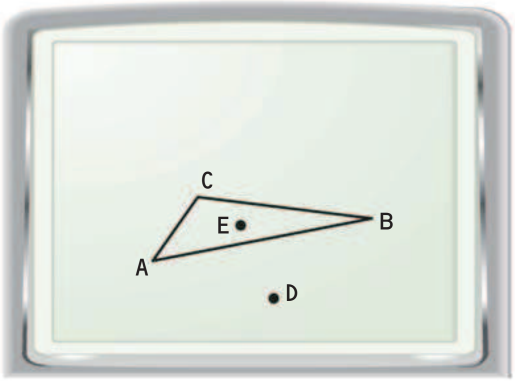
EXAMPLE 2 Draw the medians of the triangle

STEP 1



Draw medians Start with the figure you saved as "EXAMPLE1." Draw the medians of $\triangle ABC$. Label the point of concurrency E.

STEP 2

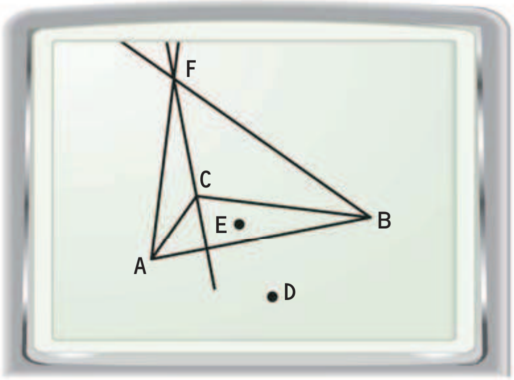


Hide the lines Use the *HIDE* feature to hide the medians. Save as "EXAMPLE2."

@HomeTutor classzone.com Keystrokes

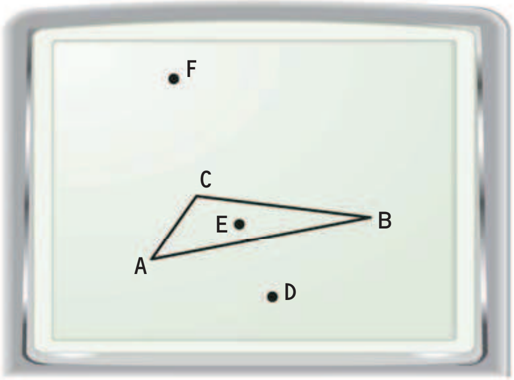
EXAMPLE 3 Draw the altitudes of the triangle

STEP 1



Draw altitudes Start with the figure you saved as "EXAMPLE2." Draw the altitudes of $\triangle ABC$. Label the point of concurrency F.

STEP 2



Hide the lines Use the *HIDE* feature to hide the altitudes. Save as "EXAMPLE3."

PRACTICE

1. Try to draw a line through points D, E, and F. Are the points collinear?

2. Try dragging point A. Do points D, E, and F remain collinear?

In Exercises 3–5, use the triangle you saved as "EXAMPLE3."

3. Draw the angle bisectors. Label the point of concurrency as point G.

4. How does point G relate to points D, E, and F?

5. Try dragging point A. What do you notice about points D, E, F, and G?

DRAW CONCLUSIONS

In 1765, Leonhard Euler (pronounced "oi′-ler") proved that the circumcenter, the centroid, and the orthocenter are all collinear. The line containing these three points is called *Euler's line*. Save the triangle from Exercise 5 as "EULER" and use that for Exercises 6–8.

6. Try moving the triangle's vertices. Can you verify that the same three points lie on Euler's line whatever the shape of the triangle? *Explain.*

7. Notice that some of the four points can be outside of the triangle. Which points lie outside the triangle? Why? What happens when you change the shape of the triangle? Are there any points that never lie outside the triangle? Why?

8. Draw the three midsegments of the triangle. Which, if any, of the points seem contained in the triangle formed by the midsegments? Do those points stay there when the shape of the large triangle is changed?