Continuing in this way, we find that the next number in the sequence, a_4, is

$$a_4 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+1}}} = 1 + \frac{1}{a_3} = 1 + \frac{1}{\frac{5}{3}} = 1 + \frac{3}{5} = \frac{8}{5}.$$

At this point, you may recognize this sequence of fractions as the ratios of consecutive pairs of Fibonacci numbers. The numbers in this sequence get closer and closer to the golden ratio ϕ, which is approximately 1.61803.

PROBLEM SET 2.3

1. Consider the sequence 1, 4, 9, 16, 25, 36, 49, 64. . . .
   a. What are the values of a_2 and a_3?
   b. What is the value of $a_4 + a_6$?
   c. If $n = 3$, what are the values of a_{n-2} and a_{n-1}?
   d. If $n = 8$, find $a_{n-3} + a_{n-1}$.
   e. If $a_n = n^2$, find the 11th, 15th, and 20th numbers in the sequence.

2. Consider the sequence 1, 8, 27, 64, 125, 216, 343, 512. . . .
   a. What are the values of a_4 and a_7?
   b. What is the value of $a_2 + a_6$?
   c. If $n = 5$, what are the values of a_{n-2} and a_{n-1}?
   d. If $n = 9$, find $a_{n-3} + a_{n-1}$.
   e. If $a_n = n^3$, find the 10th, 18th, and 25th numbers in the sequence.

3. The recursion rule, $a_n = a_{n-1} + 4$, is used to generate the numbers of a sequence.
   a. What is the rule for finding a_5?
   b. If $a_7 = 35$, what are the values of a_8, a_9, a_{10}, and a_{11}?
   c. If $a_3 = 28$, what are the values of a_2 and a_1?
   d. If $a_1 = 7$, list the first 10 numbers in the sequence.

4. The recursion rule, $a_n = 2a_{n-1} + 7$, is used to generate the numbers of a sequence.
   a. What is the rule for finding a_{10}?
   b. If $a_5 = 121$, what are the values of a_6, a_7, a_8, and a_9?
   c. If $a_3 = 77$, what are the values of a_2 and a_1?
   d. If $a_1 = 2$, list the first 10 numbers in the sequence.

5. A sequence is defined using the recursion rule $a_n = a_{n-1} \times a_{n-2}$. If $a_1 = 1$ and $a_2 = 2$, list the first 10 numbers in the sequence.

6. A sequence is defined using the recursion rule $a_n = a_{n-1} - a_{n-2}$. If $a_1 = 1$ and $a_2 = 2$, list the first 10 numbers in the sequence.

7. Find the value of the 26th Fibonacci number if the 25th Fibonacci number is 75,025 and the 27th is 196,418.

8. Find the value of the 34th Fibonacci number if the 35th Fibonacci number is 9,227,465 and the 33rd is 3,524,578.

9. Consider the first 20 numbers in the Fibonacci sequence.
   a. For each of the Fibonacci numbers from f_1 to f_{20}, determine which numbers are divisible by 3. Describe how you can tell which Fibonacci numbers will be divisible by 3 without looking at the Fibonacci number.
   b. Which of the following Fibonacci numbers will be divisible by 3: $f_{48}, f_{75}, f_{196}, f_{379}, f_{1000}$?

10. Consider the first 20 numbers of the Fibonacci sequence.
    a. Which of the Fibonacci numbers from f_1 to f_{20} are odd? Which are even? Describe how you can tell which Fibonacci numbers will be odd and which will be even without looking at the Fibonacci number.
    b. Which of the following Fibonacci numbers will be odd and which will be even: $f_{34}, f_{61}, f_{100}, f_{150}, f_{200}$?

Problems 11 and 12

The recursion rule for finding Fibonacci numbers is awkward to use if you want to find, for example, the value of 80th number in the sequence, since it requires knowledge of the values of the previous two numbers, which in turn requires knowledge of the values of the previous two numbers, and so on. There is another formula that generates the values of the Fibonacci numbers and does not depend on knowing the value of any other Fibonacci number. It is called **Binet's formula**, and it is written explicitly in terms of n, where f_n is the nth Fibonacci number.

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

11. **a.** By letting $n = 1, 2, 3,$ and 4, verify that Binet's formula generates the first four numbers in the Fibonacci sequence, namely 1, 1, 2, and 3. Use your calculator. Do not use rounded numbers while performing the calculations. Let your calculator keep all possible digits.

b. Use Binet's formula to find the value of the 40th Fibonacci number. Use your calculator, and do not use rounded numbers while performing the calculations. Let your calculator keep all possible digits.

12. **a.** Verify that the values of the 6th, 7th, 8th, and 9th Fibonacci numbers are 8, 13, 21, and 34, respectively, by using Binet's formula. Use your calculator. Do not use rounded numbers while performing the calculations. Let your calculator keep all possible digits.

b. Use Binet's formula to find the value of the 44th Fibonacci number. Use your calculator. Do not use rounded numbers while performing the calculations. Let your calculator keep all possible digits.

13. The seeds of a sunflower spiral out in two directions. A portion of each spiral in one direction is highlighted in green. A portion of each spiral in the other direction is highlighted in red. Count the spirals in each direction on the sunflower. Which Fibonacci numbers are represented by the spirals on this sunflower?

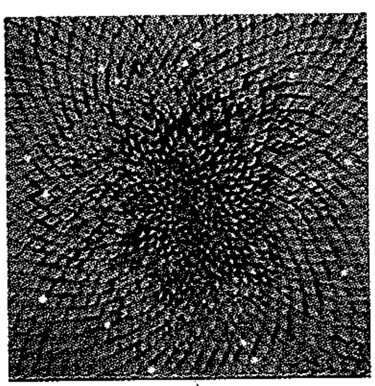
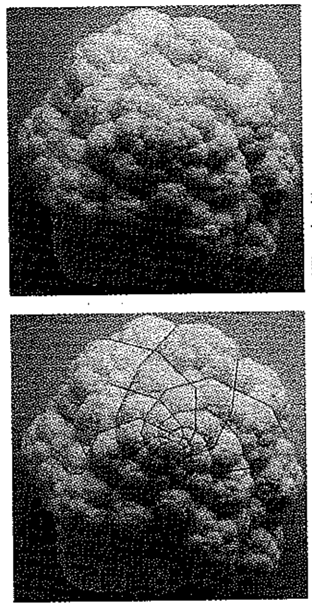


Image courtesy of http://www.goldennumber.net. Gary B. Meisner. © 2005.

14. Take a careful look at the center of a cauliflower and notice how the florets spiral. Each spiral begins in the center and moves outward. Some florets spiral clockwise, and some spiral counterclockwise. Fibonacci numbers show up in the spirals of the cauliflower.



www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html#veg

a. How many spirals extend from the center in a clockwise curve? Is it a Fibonacci number?

b. How many spirals extend from the center in a counterclockwise curve? Is it a Fibonacci number?

15. In his book *536 Puzzles and Curious Problems*, Henry Dudeney posed a problem involving cows and reproduction. Suppose a cow produces its first female calf at the age of 2 years and then produces another single female calf every year. Suppose we begin with one female calf in its first year. Assume none die.

a. What is the total number of cows and calves for each of the first 10 years?

b. How many adult cows are there in each of the first 10 years?

c. How many calves are there in each of the first 10 years?

d. Use your observations to predict the number of cows and calves in each of the years 11 through 15.

16. In the 14th century, an Indian mathematician named Narayana wondered what would happen if the ability of a cow to reproduce was delayed. Suppose each cow produces one female calf each year beginning in its fourth year, at which time it is considered an adult. Suppose we begin with one female calf in its first year and none die.

a. What is the total number of cows and calves for each of the first 11 years?

b. How many adult cows are there each year for each of the first 11 years?

c. How many calves are there each year for each of the first 11 years?

d. While the total number of cows and calves each year does not form a Fibonacci sequence, a similar recursion rule will generate the sequence. Find the rule and predict the number of cows and calves in each of the years 12 through 16.

Problems 17 and 18

A sequence of numbers similar to the Fibonacci sequence, called the Lucas sequence, uses the same recursion rule but different starting numbers. The rule for the Lucas sequence is $L_n = L_{n-1} + L_{n-2}$ where $L_1 = 1$ and $L_2 = 3$.

17. a. Give the first 10 numbers in the Lucas sequence.

b. For the Fibonacci sequence defined by $f_n = f_{n-1} + f_{n-2}$ with $f_1 = 1$ and $f_2 = 1$, and the Lucas sequence defined by the recursion rule $L_n = L_{n-1} + L_{n-2}$ with $L_1 = 1$ and $L_2 = 3$, complete the following table. What do you notice about the numbers in the last two columns?

n	f_n	L_n	$f_n \times L_n$	f_{2n}
1				
2				
3				
4				
5				

18. a. Given that $L_2 = 3$, find the next 10 values of the Lucas sequence by using the numbers in the Fibonacci sequence according to the recursion rule $L_n = f_{n+1} + f_{n-1}$.

b. For the Fibonacci sequence defined by the recursion rule $f_n = f_{n-1} + f_{n-2}$ with $f_1 = 1$ and $f_2 = 1$ and the Lucas sequence defined by the recursion rule $L_n = L_{n-1} + L_{n-2}$ with $L_1 = 1$ and $L_2 = 3$, complete the following table. What do you notice about the numbers in the last two columns?

n	f_n	L_n	$\frac{f_n + L_n}{2}$	f_{n+1}
1				
2				
3				
4				
5				

Problems 19 and 20

Fibonacci numbers are generated using a recursion rule that relates each number to the two numbers that precede it. Similar sequences can be generated by using more than two numbers to determine a number in the sequence.

19. The Tribonacci numbers can be generated using the recursion rule $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, where $T_1 = 1$, $T_2 = 1$, and $T_3 = 2$.

a. Find the first 12 Tribonacci numbers.

b. Find the ratios of consecutive pairs of Tribonacci numbers from part (a); that is, calculate $\frac{T_2}{T_1}, \frac{T_3}{T_2}, \frac{T_4}{T_3}, \ldots, \frac{T_{12}}{T_{11}}$. What do you notice about the ratios?

20. The Tetranacci numbers can be generated using the recursion rule $T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$, where $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and $T_3 = 4$.

a. Find the first 12 Tetranacci numbers.

b. Find the ratios of consecutive pairs of Tetranacci numbers from part (a); that is, calculate $\frac{T_2}{T_1}, \frac{T_3}{T_2}, \frac{T_4}{T_3}, \ldots, \frac{T_{12}}{T_{11}}$. What do you notice about the ratios?

Problems 21 through 24

Consider what would happen to the Fibonacci sequence if, rather than adding the previous two numbers to calculate the next number, we either add or subtract the two preceding numbers according to the result of a coin toss. In 1999, Divakar Viswanath studied this idea of adding an element of randomness to the Fibonacci sequence. Suppose the first two numbers in this random Fibonacci sequence are $V_1 = 1$ and $V_2 = 1$. To determine each successive number, a coin is flipped. If the result of the flip is heads, the next number will be the sum of the previous two numbers—that is, $V_n = V_{n-1} + V_{n-2}$. If the result is tails, then the next number will be the difference of the previous two numbers, that is, $V_n = V_{n-2} - V_{n-1}$.

21. a. Given that the first two numbers in the random Fibonacci sequence are $V_1 = 1$ and $V_2 = 1$, find the next 8 numbers in the sequence if coin tosses result in the alternating pattern HTHTHTHT.

b. What sequence results if the coin always comes up heads?

22. a. Given that the first two numbers in the random Fibonacci sequence are $V_1 = 1$ and $V_2 = 1$, find the next 8 numbers in the sequence if coin tosses result in the alternating pattern THTHTHTH.

b. What sequence results if the coin always comes up tails?

23. The first nine numbers in a random Fibonacci sequence are 1, 1, 0, 1, 1, 0, 1, 1, 0. What pattern of heads and tails generated the 3rd through 9th numbers in the sequence?

24. The first 12 numbers in a random Fibonacci sequence are 1, 1, 2, 3, −1, 2, 1, 1, 2, 3, −1, 2. What pattern of heads and tails generated the 3rd through 12th numbers in the sequence?

Problems 25 and 26

The quadratic formula gives the solutions for a quadratic equation of the form $ax^2 + bx + c = 0$. Solutions to this equation are given by the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

25. Pell numbers are generated by the recursion rule $P_n = 2P_{n-1} + P_{n-2}$ where $P_1 = 1$ and $P_2 = 2$.

a. Find the first 12 Pell numbers.

b. Find the ratios of consecutive pairs of Pell numbers from part (a); that is, calculate $\frac{P_2}{P_1}, \frac{P_3}{P_2}, \frac{P_4}{P_3}, \ldots, \frac{P_{12}}{P_{11}}$. What do you notice about the ratios?

c. Use the quadratic formula to find solutions to the quadratic equation $x^2 - 2x - 1 = 0$ and compare your answers to the quotients from part (b). What do you notice?

26. Consider the sequence generated by the recursion rule $G_n = 2G_{n-1} + 2G_{n-2}$, where $G_1 = 1$ and $G_2 = 2$.

a. Find the first 12 numbers in this sequence.

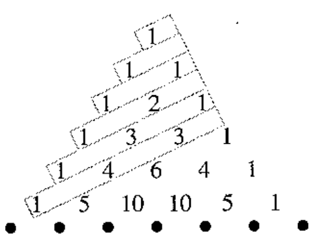
b. Find the ratios of consecutive pairs of numbers in the sequence from part (a); that is, calculate $\frac{G_2}{G_1}, \frac{G_3}{G_2}, \frac{G_4}{G_3}, \ldots, \frac{G_{12}}{G_{11}}$. What do you notice about the ratios?

c. Use the quadratic formula to find solutions to the quadratic equation $x^2 - 2x - 2 = 0$ and compare your answers to the ratios you found in part (b). What do you notice?

27. Pascal's triangle, shown next, is a triangular array of numbers in which each entry other than a 1 is obtained by adding the two entries in the row immediately above it. For example, the first 3 in row 4 of the triangle is found by adding $1 + 2 = 3$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
• • • • • •

Find the sums of the numbers on the diagonals in Pascal's triangle as shown next. Do you see a pattern? Explain.

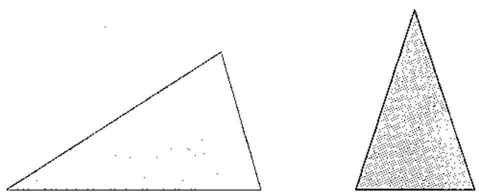


28. Predict the sums along the next four diagonals in the previous problem. Check your answer by adding new rows to Pascal's triangle and adding the entries along the diagonals.

29. Recall that the golden ratio is denoted by ϕ, where $\phi = \dfrac{1+\sqrt{5}}{2}$. A **golden triangle** is an isosceles triangle (a triangle with two sides of equal length) such that

$$\frac{\text{long side length}}{\text{short side length}} = \phi .$$

a. Measure the lengths of the sides of the following triangles and determine if either of the triangles is a golden triangle.



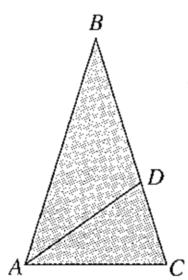
(i) (ii)

b. Determine the length of the long side of a golden triangle if the short side is ϕ units.

c. Find the length of the short side of a golden triangle if the long side is 1 inch.

d. Find the length of the short side of a golden triangle if the long side is 7 cm.

30. In the following figure, both $\triangle ABC$ and $\triangle ADC$ are golden triangles. (See the definition of a golden triangle in the previous problem.)



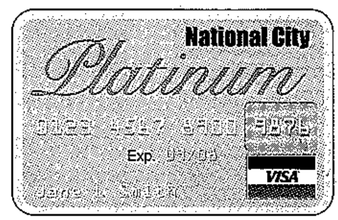
a. If $AC = 1$ unit, then find AB, DC, and BD.

b. Use the result from part (a) to show that $\triangle ABD$ is also a golden triangle, that is, show that $\dfrac{AB}{BD} = \phi$.

(*Hint:* Use the fact that $\phi = \dfrac{1+\sqrt{5}}{2}$.)

31. Use the given measurements to determine which of the following more closely approximates a golden rectangle. Justify your response.

a. Credit card (54 mm by 86 mm)



b. Post Alpha-Bits box (8 inches by 11.75 inches)

32. Use the given measurements to determine which of the following more closely approximates a golden rectangle. Justify your response.

a. Nabisco Cream of Wheat box (13.5 mm by 19.8 mm)

b. Rice A Roni box ($3\frac{3}{4}$ inches by $6\frac{5}{16}$ inches)

33. If a golden rectangle has its shorter side of length 11 mm, what is the length of its longer side?

34. If a golden rectangle has its longer side of length of 9.25 in., what is the length of its shorter side?