SOLUTION OF THE INITIAL PROBLEM

The Compromise of 1850 averted civil war in the United States for 10 years. This compromise began as a group of eight resolutions presented as a single bill to the Senate on January 29, 1850, by Henry Clay of Kentucky. Six months of speeches, bargaining, and amendments culminated in the defeat of Clay's measure on July 31, 1850. Yet shortly thereafter, essentially the same proposals were shepherded to passage by Stephen Douglas of Illinois. How is this possible?

SOLUTION Within any set of voting rules, if a sufficiently large group of voters is presented with more than two alternatives, almost anything can happen. At the beginning of 1850, 30 states were in the Union. Thus, there were 60 Senators, hence plenty of voters. Clay's eight resolutions included many issues, and he attempted to get a majority of the senators to accept all the resolutions in one bill. Clay found that this was impossible, so he tried to find packages of resolutions that could be passed with a majority. However, the parliamentary maneuvering needed to package, repackage, and amend the proposals had a divisive and negative effect. So many senators (sometimes northern, sometimes southern) disagreed with particular resolutions that Clay could not package the resolutions in such a way that all could pass.

As combined by Clay, the measures could not pass, but Douglas was able to build a majority for each *individual* piece of the legislation. Each individual bill was then passed with a majority, and the entire compromise went into effect. Some votes were very close. For example, only 32 senators voted to admit California as a free state. It is interesting to note that only 4 senators of the 60 voted in favor of all the bills, and many senators did not vote on the pieces of the legislation that were unpopular in their home states. Even though individually each piece of legislation passed by a majority, taken together, the collection of eight resolutions never received a majority, a quirk of voting that is still surprising today.

PROBLEM SET 3.2

1. Consider the following preference table of nine voters with three choices:

Ranking	Number of Voters			
	4	3	1	1
1st	A	C	B	A
2nd	C	B	C	B
3rd	B	A	A	C

a. If a candidate is required to win an election by receiving a majority of first-place votes, who wins this election?

b. Who wins the election using the Borda count method?

c. Identify which criterion has been violated.

2. Consider the following preference table of 11 voters with four choices:

Ranking	Number of Voters							
	2	2	2	1	1	1	1	1
1st	B	C	B	C	C	B	A	B
2nd	A	D	C	B	D	D	C	C
3rd	C	A	A	A	B	C	D	D
4th	D	B	D	D	A	A	B	A

a. If a candidate is required to win an election by receiving a majority of first-place votes, who will win this election?

b. Who will win the election using the Borda count method?

c. Identify which criterion has been violated.

3. Nine city council members vote on three renovation proposals, A, B, and C, for the city center. Consider the members' preferences as given in the following table.

Ranking	Number of Members			
	3	3	2	1
1st	C	A	B	A
2nd	B	C	C	B
3rd	A	B	A	C

a. Which plan is adopted under the plurality method?

b. Form all pairwise comparisons and determine the winner of each.

c. Consider the results from parts (a) and (b), and identify which criterion has been violated.

4. Eleven school board members vote on four cost-reduction plans involving the district bussing service. The board members' preferences for plans A, B, C, and D are summarized in the following table.

Ranking	Number of Members								
	2	2	1	1	1	1	1	1	1
1st	B	B	B	C	A	D	A	A	C
2nd	A	C	D	B	D	C	C	C	D
3rd	C	A	C	A	C	B	D	B	B
4th	D	D	A	D	B	A	B	D	A

a. Which plan is adopted under the plurality method?

b. Form all pairwise comparisons and determine the winner of each.

c. Consider the results from parts (a) and (b), and identify which criterion has been violated.

5. Consider the following preference table for 10 voters faced with three alternatives, A, B, and C. Show that, in this case, the plurality method violates the head-to-head criterion.

Ranking	Number of Voters		
	4	3	3
1st	A	B	C
2nd	B	C	B
3rd	C	A	A

6. Consider the following preference table for 13 voters faced with three alternatives, A, B, and C. Show that, in this case, the plurality method violates the head-to-head criterion.

Ranking	Number of Voters		
	5	4	4
1st	B	C	A
2nd	C	A	C
3rd	A	B	B

7. Nine faculty members interview three candidates, Jackson (J), Carter (C), and Morton (M), for a new full-time position. The preferences are listed in the following table.

Ranking	Number of Faculty				
	4	2	1	1	1
1st	J	C	C	M	M
2nd	C	M	J	C	J
3rd	M	J	M	J	C

a. Find the winner using the plurality method.

b. Find the winner using the pairwise comparison method. Is any candidate a Condorcet candidate?

c. Find the winner using the Borda count method.

d. Consider the results from parts (a), (b), and (c), and identify which criterion has been violated and by which method. Justify your answer.

8. Ten members of the high school cheerleading squad vote on three applicants, Jesse (J), Lorna (L), and Carmen (C). The preferences are listed in the following table.

Ranking	Number of Members			
	5	3	1	1
1st	J	L	C	L
2nd	L	C	J	J
3rd	C	J	L	C

a. Find the winner using the plurality method.

b. Find the winner using the pairwise comparison method. Are there any applicants who are Condorcet candidates?

c. Find the winner using the Borda count method.

d. Consider the results from parts (a), (b), and (c), and identify which criterion has been violated and by which method. Justify your answer.

9. The following table contains the preferences of 51 voters faced with four choices.

Ranking	Number of Voters					
	18	10	9	7	4	3
1st	A	C	C	B	D	D
2nd	D	B	D	D	B	C
3rd	C	D	B	C	A	A
4th	B	A	A	A	C	B

a. Find the winner using the Borda count method.

b. Find the winner using the pairwise comparison method.

c. Find the winner using the plurality with elimination method.

d. Consider the results from parts (a), (b), and (c), and identify which criterion has been violated and by which method. Justify your answer.

10. The following table contains the preferences of 25 voters faced with four choices.

Ranking	Number of Voters		
	10	10	5
1st	A	B	C
2nd	C	C	A
3rd	B	A	B
4th	D	D	D

a. Find the winner using the Borda count method.

b. Find the winner using the pairwise comparison method.

c. Find the winner using the plurality with elimination method.

d. Consider the results from parts (a), (b), and (c), and identify which criterion has been violated and by which method. Justify your answer.

11. Consider the following preference table for seven voters and three alternatives.

Ranking	Number of Voters		
	3	2	2
1st	A	B	C
2nd	B	C	B
3rd	C	A	A

a. Which alternative is selected under the plurality method?

b. Suppose the election is held again, but alternative C is removed from consideration. Which alternative is selected under the plurality method?

c. Identify which criterion has been violated and in which election. Justify your answer.

12. Consider the following preference table for 10 voters and 3 candidates.

Ranking	Number of Voters		
	4	3	3
1st	A	C	B
2nd	B	B	C
3rd	C	A	A

a. Which candidate is selected under the plurality method?

b. Suppose the election is held again, with candidate C removed from consideration. Which candidate is selected under the plurality method?

c. Identify which criterion has been violated and in which election. Justify your answer.

13. The following table contains preferences of 5 voters for 3 alternatives.

Ranking	Number of Voters	
	3	2
1st	B	A
2nd	A	C
3rd	C	B

a. Which alternative wins under the Borda count method?

b. Which alternative wins under the plurality method? Does that alternative have a plurality or a majority?

c. Identify which criterion has been violated based on the results from parts (a) and (b).

d. Which other criterion has been violated? Justify your response

14. The following table contains preferences of 7 voters for 3 alternatives.

Ranking	Number of Voters	
	3	4
1st	A	B
2nd	C	A
3rd	B	C

a. Which alternative wins under the Borda count method?

b. Which alternative wins under the plurality method? Does that alternative have a plurality or a majority?

c. Identify which criterion has been violated based on the results from parts (a) and (b).

d. Which other criterion has been violated? Justify your response.

15. Show that for the following preference table of 5 voters and 3 alternatives, the Borda count method fails to satisfy the irrelevant-alternatives criterion.

Ranking	Number of Voters	
	3	2
1st	B	A
2nd	A	C
3rd	C	B

16. Show that for the following preference table of 7 voters and 3 alternatives, the Borda count method fails to satisfy the irrelevant-alternatives criterion.

Ranking	Number of Voters	
	3	4
1st	A	B
2nd	C	A
3rd	B	C

17. Show that for the following preference table of 5 voters faced with 4 alternatives, the plurality with elimination method fails to satisfy the head-to-head criterion.

Ranking	Number of Voters		
	2	2	1
1st	A	B	C
2nd	D	D	D
3rd	B	C	A
4th	C	A	B

18. Show that for the following preference table of 7 voters faced with 4 alternatives, the plurality with elimination method fails to satisfy the head-to-head criterion.

Ranking	Number of Voters		
	3	3	1
1st	A	B	C
2nd	D	D	D
3rd	B	A	A
4th	C	C	B

19. Suppose 17 faculty members on a scholarship committee rank 3 applicants. The preferences are given in the following table.

Ranking	Number of Faculty			
	6	5	4	2
1st	A	B	C	C
2nd	C	A	B	A
3rd	B	C	A	B

a. Which candidate wins the scholarship under the plurality with elimination method?

b. Suppose the two faculty members who preferred C to A to B changed their ballots at the last minute to show they preferred A to C to B, thus benefiting candidate A. Which candidate wins the scholarship under the plurality with elimination method?

c. Identify which criterion has been violated.

20. Twenty-one judges rank three contestants in a talent contest. The following table contains the preferences of the judges.

Ranking	Number of Judges				
	8	5	4	2	2
1st	C	A	~~B~~	A	~~B~~
2nd	~~B~~	~~B~~	C	C	A
3rd	A	C	A	~~B~~	C

A 7 ~~B~~ C 8

a. Which contestant wins under the plurality with elimination method?

b. Some of the judges decided to change their preferences, and the results of those changes are listed in the following table. Describe the changes, and determine the winner using the plurality with elimination method.

C = 12
A 9

Ranking	Number of Judges				
	8	5	4	2	2
1st	C	~~A~~	B	C	B
2nd	B	B	C	~~A~~	~~A~~
3rd	~~A~~	C	~~A~~	B	C

C 10 ~~A 5~~ B 6

c. Identify which criterion has been violated.

C 10 B 11

21. Nine voters rank three candidates, and their preferences are given in the following table.

Ranking	Number of Voters		
	4	3	2
1st	A	B	C
2nd	B	C	A
3rd	C	A	B

a. Who wins under the plurality method?

b. Who wins under the plurality with elimination method?

c. Suppose candidate B has to remove himself from the ballot. Create a new preference table and determine the winner under each method.

d. Identify which criterion has been violated and by which method.

22. Twelve voters rank three candidates, and their preferences are given in the following table.

Ranking	Number of Voters		
	5	4	3
1st	C	A	B
2nd	A	B	C
3rd	B	C	A

a. Who wins under the plurality method?

b. Who wins under the plurality with elimination method?

c. Suppose candidate A has to remove herself from the ballot. Create the new preference table and determine the winner under each method.

d. Identify which criterion has been violated and by which method.

23. Faculty members vote on four different reading programs for use in the district's elementary schools. The preferences are listed in the following table.

Ranking	Number of Faculty				
	12	10	7	6	4
1st	B	A	D	C	A
2nd	C	D	C	A	D
3rd	D	C	A	D	C
4th	A	B	B	B	B

a. Is there a Condorcet program? Explain.

b. Which program wins under the plurality method? Does this case violate the head-to-head or irrelevant-alternatives criterion? Justify your responses.

c. Which program wins under the Borda count method? Does this case violate the majority or head-to-head criterion in this case? Justify your responses.

24. A 28-member Board of Directors votes to decide which of 5 charities will receive a large donation. The preferences are given in the following table.

Ranking	Number of Voters						
	7	5	5	5	3	2	1
1st	B	B	D	E	A	C	C
2nd	A	A	A	C	D	B	D
3rd	C	D	E	D	C	A	A
4th	D	E	C	B	B	D	B
5th	E	C	B	A	E	E	E

a. Is there a Condorcet charity? Explain.

b. Which charity wins under the pairwise comparison method? Does this case violate the irrelevant-alternatives criterion? Justify your response.

c. Which charity wins under the Borda count method? Does this case violate the head-to-head or the irrelevant-alternatives criterion? Justify your responses.

25. Consider the following preference table for 12 voters who ranked 3 candidates.

Ranking	Number of Voters		
	5	4	3
1st	C	A	B
2nd	A	B	C
3rd	B	C	A

a. What is the result under the Borda count method?

b. Suppose a variation of the Borda count method, called **Nanson's method**, is used in which a standard Borda count is generated in the first round. The candidate receiving the lowest Borda count is eliminated, and the count is recalculated. These steps are repeated until one candidate is left. Who is the winner under Nanson's method?

c. This preference table demonstrates that Nanson's method does not always satisfy a certain criterion. Which criterion does it violate? Justify your answer.

26. In the **Bucklin method**, all the first-place votes are counted. If one candidate receives a majority, he or she wins. If no candidate has a majority, then the first- and second-place votes for each candidate are counted. If no candidate has a majority, then the first-, second-, and third-place votes are counted. This process is continued until a majority winner is found. If more than one candidate has a majority in a round, the winner is the candidate with the greatest total. Consider the following preference table.

Ranking	Number of Voters					
	18	10	9	7	4	3
1st	A	C	C	B	D	D
2nd	D	B	D	D	B	C
3rd	C	D	B	C	A	A
4th	B	A	A	A	C	B

a. Which candidate wins under the Bucklin method?

b. Determine whether the Bucklin method violates the monotonicity criterion in this case. Justify your response.

c. Determine whether the Bucklin method violates the head-to-head criterion in this case. Justify your response.

d. Determine whether the Bucklin method violates the majority criterion in this case. Justify your response.

Problems 27 and 28

At the state convention on May 11, 2002, the Utah Republican Party used instant runoff voting to nominate candidates to the U.S. House of Representatives. The delegates ranked the candidates using a preferential ballot. In the first round, the candidate with the lowest number of first-place votes was eliminated and his or her votes were redistributed. This continued round by round until one candidate achieved at least 60% of the vote or until there were two candidates left, in which case they would face each other in a primary election in June.

27. Consider the following summary of each round of the process for Utah Congressional District 1.

	Number of First-Place Votes								
Candidate	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	Round 8	Round 9
Bishop	304	304	304	304	306	340	388	473	602
Garn	189	189	189	192	193	221	257	331	432
McCall	182	182	182	183	183	186	204	239	
Wyatt	136	137	137	139	140	168	194		
Gross	111	111	112	113	116	133			
Probasco	100	100	100	103	110				
Jacobs	13	13	13	14					
Lee	11	11	11						
Sundwall	1	1							
Barker	1								

a. How many delegates voted in this election?

b. Who were the first three candidates to be eliminated? For each of the ballots that were redistributed in those first three rounds, determine and list the first- and second-place candidates.

c. Despite the fact that 10 candidates ran for office, not every voter ranked every candidate. Determine from the table the fewest number of rankings provided by a voter and the total number of voters who failed to rank every candidate.

d. Suppose the plurality method had been used instead. Who would have won in round 1? Now suppose that Wyatt was disqualified and that all ballots listing Wyatt as the first choice also listed McCall as the second choice. What would the plurality method's outcome be in this case, and what fairness criterion would be violated?

28. Consider the following summary of each round of the process for Utah Congressional District 2.

Candidate	Number of First-Place Votes										
	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	Round 8	Round 9	Round 10	Round 11
Bridgewater	246	248	248	248	251	256	273	295	358	471	653
Swallow	276	276	278	279	279	285	292	318	348	401	530
Jorgensen	170	170	170	171	172	179	195	220	251	317	
Wilkinson	134	138	141	147	151	156	169	192	238		
Stephens	146	147	147	149	151	154	160	171			
Cook	94	94	96	98	102	106	109				
Crockett	51	51	51	52	56	62					
Harmsen	35	35	35	35	36						
Snelgrove	16	16	16	19							
Smith	13	13	16								
Howard	10	10									
Towner	7										

a. How many delegates voted in this election?

b. Who were the first two candidates to be eliminated? Create a list of the first- and second-place candidates for each of the ballots that were redistributed in those first two rounds.

c. Despite the fact that 12 candidates ran for office, not every voter ranked every candidate. Determine from the table the fewest number of rankings provided by a voter and the total number of voters who failed to rank every candidate.

d. Suppose the plurality method had been used. Who would have won in round 1? Now suppose that Harmsen was disqualified and that all the ballots listing Harmsen as the first choice also listed Bridgewater as the second choice. What would the plurality method's outcome be in this case, and what fairness criterion would be violated?

29. On April 1, 2003, voters in Henry County, Illinois, used approval voting to fill three Village Trustee positions for the Village of Alpha. If the following ballots were cast, which three candidates were elected?

Candidates	Number of Ballots							
	103	70	45	42	7	2	1	1
Althaus	X			X	X			
Wirt	X	X	X					
Medley	X			X		X		
Brown		X	X	X		X	X	X
Kofoid		X			X		X	

30. Suppose the board of directors of the United Way had used approval voting to fill two vacancies on the executive committee, and the 25 ballots had been marked as follows. Which two candidates would win seats on the executive committee?

	Ballots																									
A	X		X		X	X	X		X	X		X			X		X		X			X	X	X	X
B	X	X	X	X	X	X		X	X				X	X		X			X	X	X		X		X
C		X			X			X	X	X		X		X			X	X			X	X			X
D	X		X		X		X				X			X	X	X		X	X		X			X	

Problems 31 and 32

The Association for the Advancement of Automotive Medicine, an organization dedicated to motor vehicle crash injury prevention and control, uses approval voting to elect members to their board of directors. Prior to each annual meeting, five directors must be elected, two of whom may be professionals other than physicians. Association members elect three directors from one group of four candidates and two directors from another group of four candidates.

31. Suppose voters return the following ballots. Which three candidates are elected to the board of directors?

Candidates	Number of Voters			
	580	423	244	126
Deegear	X		X	X
Maio		X	X	X
Soderstrom	X	X		X
Vaca	X	X	X	

32. Suppose the voters return the following ballots. Which two candidates are elected to the board of directors?

Candidates	Number of Voters					
	529	321	207	198	65	53
Kent	X		X	X		
Langwieder			X		X	X
McCartt	X	X			X	
Pintar		X		X		X

33. The high school senior class will host an all-night graduation party. The class will use approval voting to decide which two main dishes will be served. The following table contains the vote results.

Main Dishes	Number of Students					
	17	25	11	9	13	2
Pizza	X		X			
Tacos		X	X		X	
Hamburgers	X		X	X		
Chinese food	X	X	X		X	
Sub sandwiches	X	X	X			

a. Which two main dishes will be served at the party?

b. Eleven students voted for all the options. How would the results change if those 11 students had left their ballots completely blank? How can blank ballots be interpreted?

34. a. Suppose in the previous problem, five more ballots appeared, each indicating tacos as the only choice. Which two main courses would be served at the party?

b. One argument in favor of approval voting involves the idea that the voters are more able to express their true feelings with an approval ballot. Interpret the meaning of each column of ballots originally cast in the previous problem.

Problems 35 through 38

The All Natural Juice Company conducted taste tests to determine which of four new flavors to market. The testers tasted each juice and ranked them in order of preference, as shown in the following table.

Ranking	Number of Testers				
	13	12	10	8	7
1st	B	D	A	C	C
2nd	A	A	B	D	B
3rd	D	B	C	B	A
4th	C	C	D	A	D

35. Identify which flavor wins the taste test using the plurality method, and determine whether this method violates any of the four fairness criteria. If a violation occurs, explain why.

36. Identify which flavor wins the taste test using the plurality with elimination method, and determine whether this method violates any of the four fairness criteria. If a violation occurs, explain why.

37. Identify which flavor wins the taste test using the Borda count method, and determine whether this method violates any of the four fairness criteria. If a violation occurs, explain why.

38. Identify which flavor wins the taste test using the pairwise comparison method, and determine whether this method violates any of the four fairness criteria. If a violation occurs, explain why.

Extended Problems

39. During the 2000 Presidential election, attention was focused on spoiled ballots. In Florida, close attention was given to "undervotes" and "overvotes" as ballots were studied to see if the voter's intent could be determined. Research these terms and describe the various ways a ballot could be spoiled. Suppose approval voting had been used instead. Would these ballots still be considered spoiled? Explain.

40. Explain why the plurality method must satisfy the monotonicity criterion regardless of the number of voters or alternatives. You might start by creating a few preference tables and then trying to create one for which the method does not satisfy the criterion.

41. Explain why any method that violates the majority criterion must also violate the head-to-head criterion. You might begin by examining a few preference tables for which the majority criterion is violated.

42. We have concentrated on voting methods in which voters have more than two choices. For the special case in which an odd number of voters vote for two alternatives, the only voting system that satisfies the monotonicity criterion is one in which a majority is required for a win. Two other desirable properties are summarized in a theorem, attributed to Kenneth May in 1952, called May's theorem. Research May's theorem and summarize your findings in a report. Search keywords "May's theorem" on the Internet.

43. Theodore Roosevelt lost the Republican nomination in 1911 and created the Progressive Party, also known as the Bull Moose Party, so he could run on his own third-party ticket. He split the Republican vote, allowing Woodrow Wilson, the Democratic candidate, to win. As a result, several states created "sore loser" laws to prevent losing candidates from forming third parties. Research "sore loser" laws. Does your state have such a law? What barriers in your state make it more difficult for third-party candidates to access the ballot? Write a report to summarize your findings.

44. Historians hypothesize that Abraham Lincoln would never have become our 16th president if a different voting method had been used. Four candidates were on the ballot in the 1860 election: Abraham Lincoln, Stephen Douglas, John Breckinridge, and John Bell. Lincoln won by a plurality, with 40% of the popular vote. He received no votes in seven southern states. By the time he was inaugurated, seven states had left the union and the Civil War was about to begin. Read about the 1860 election. Research the popularity of each candidate. Where did the support for each candidate come from? Suppose the approval method of voting had been used to elect the president instead. Using your research, create a table giving a possible scenario for votes cast using the approval method in the 1860 election. Under this method, who would have won the election and how might history have changed? Write a report to summarize your findings.

45. Which U.S. presidents have won the presidency by a plurality rather than a majority? Which presidents have won the popular vote yet lost because of the electoral college vote? Do you think it would be fairer to elect the U.S. President by plurality voting, by approval voting, or by the electoral system that is currently used? Justify your response.

Problems 46 and 47

In addition to the shortcomings we discussed in this section, several of the voting methods can be manipulated through insincere voting or other means. Insincere voting (not voting in accordance with your true preferences) may be practiced to gain long-run strategic advantage, such as during a sequence of runoffs. For instance, if you know a candidate you really dislike is poised to win, you might change your vote to support a