Table 5.30

Heir and Number of Hours Worked	Standard Quota	Rounded Standard Quota	Modified Quota	Rounded Modified Quota
You, 72	40.91	41	$\frac{72}{1.77} \approx 40.68$	41
Your sister, 43.5	24.72	25	$\frac{43.5}{1.77} \approx 24.58$	25
Your brother, 34.5	19.60	20	$\frac{34.5}{1.77} \approx 19.49$	19

Now the sum of the rounded modified quotas is $41 + 25 + 19 = 85$, which is exactly what we want. Thus, by Webster's method, you would receive 41 coins, your sister 25, and your brother 19.

Notice that under each of these methods you received 41 coins, so neither method is more advantageous for you.

PROBLEM SET 5.2

1. If you want the value of a fraction to increase while leaving the numerator alone, how must the denominator change?

2. If you want the value of a fraction to decrease while leaving the numerator alone, how must the denominator change?

3. For each of the following pairs of fractions, insert the symbol > or < between the fractions to make a true statement. Do not use your calculator.
   a. $\frac{500}{6} \quad \frac{500}{7}$
   b. $\frac{7500}{37} \quad \frac{7500}{37 + 1}$
   c. $\frac{2{,}345{,}674}{357 + 2} \quad \frac{2{,}345{,}674}{357 + 1}$

4. For each of the following pairs of fractions, insert the symbol > or < between the fractions to make a true statement. Do not use your calculator.
   a. $\frac{3440}{15} \quad \frac{3440}{14}$
   b. $\frac{12{,}433}{85} \quad \frac{12{,}433}{85 + 1}$
   c. $\frac{6{,}001{,}230}{276 + 3} \quad \frac{6{,}001{,}230}{276 + 2}$

5. While using Jefferson's method, you realize that modified quotas must be created that are larger than the standard quotas. To accomplish this, you must modify the divisor. How will the modified divisor compare to the standard divisor?

6. While using Jefferson's method, you realize that your choice of a modified divisor makes your modified quotas too large. Will your next choice of modified divisor be larger or smaller than the modified divisor you just used? Explain.

7. Suppose Jefferson's method is going to be used to apportion 27 school board seats to 5 zones.
   a. If the modified quotas have been determined to be 6.97, 8.71, 4.65, 3.48, and 3.19, what should your next step be?
   b. If the modified quotas have been determined to be 7.95, 9.93, 5.30, 3.97, and 3.64, what should your next step be?

8. Suppose Jefferson's method will be used to apportion 60 shares of stock to 6 beneficiaries.
   a. If the modified quotas have been determined to be 7.55, 15.85, 3.77, 26.04, 4.91, and 5.28, what should your next step be?
   b. If the modified quotas have been determined to be 7.81, 16.41, 3.91, 26.95, 5.08, and 5.47, what should your next step be?

9. Suppose 6 seats will be apportioned to three states, A, B, and C, with populations of 101, 109, and 150, respectively, according to Jefferson's method.

a. Find the total population, the standard divisor and the standard quotas. Do the integer parts of the standard quotas add up to the number of available seats?

b. A modified divisor must be created. How will the modified divisor compare with the standard divisor? Select a modified divisor and explain the rationale behind the selection.

c. Fill in the following table using the given modified divisors.

Modified Divisor	Modified Quota State A	Modified Quota State B	Modified Quota State C	Sum of Integer Parts
$d = 50$				
$d = 51$				
$d = 50.5$				

d. How should the 6 seats be apportioned to the states?

10. Suppose 5 seats will be apportioned to three states, A, B, and C, with populations of 525, 610, and 650, respectively, according to Jefferson's method.

a. Find the total population, the standard divisor, and the standard quotas. Do the integer parts of the standard quotas add up to the number of available seats?

b. A modified divisor must be created. How will the modified divisor compare with the standard divisor? Select a modified divisor and explain the rationale behind the selection.

c. Fill in the following table using the given modified divisors.

Modified Divisor	Modified Quota State A	Modified Quota State B	Modified Quota State C	Sum of Integer Parts
$d = 325$				
$d = 217$				
$d = 271$				

c. How should the 5 seats be apportioned to the states?

11. Three friends have pooled their resources to bid on 20 bottles of vintage red wine at a wine auction. They decide to divide the bottles based on the amount each person contributed, using Jefferson's method of apportionment. Jaron contributed \$295, Mikkel contributed \$205, and Robert contributed \$390.

a. Fill in the following table, and add rows as needed. Show how you arrived at each modified divisor used.

Contribution	Jaron \$295	Mikkel \$205	Robert \$390	Sum of Integer Parts
Standard quota				
Modified quota				
Modified quota				
Modified quota				

b. How should the bottles be apportioned among the friends?

12. From problem 11, suppose another 5 bottles of the wine were added to the lot, bringing the total to 25 bottles. In addition, a fourth friend, Monica, agrees to chip in \$180. These four friends decide to divide the bottles based on the amount each person contributed using Jefferson's method of apportionment.

a. Fill in the following table, and add rows as needed. Show how you arrived at each modified divisor used.

Contribution	Jaron \$295	Mikkel \$205	Robert \$390	Monica \$180	Sum of Integer Parts
Standard quota					
Modified quota					
Modified quota					
Modified quota					

b. How should the bottles be apportioned?

13. The faculty senate at Dartvard University decided to reorganize in 2001. The new 30-seat senate would be apportioned according to Jefferson's method and be based on enrollment in its five colleges. Apportion the faculty senate according to Jefferson's method. Organize your work in a table.

College	Enrollment
Fine and Performing Arts	2540
Math and Physical Science	3580
Engineering	1410
Social Science	1830
Agriculture	750

14. Continuing from problem 13, the faculty senate at Dartvard University reorganizes every 3 years. Apportion the 30 seats in the 2004 faculty senate according to Jefferson's method and new 2004 enrollment figures. Organize your work in a table.

College	Enrollment
Fine and Performing Arts	2930
Math and Physical Science	3320
Engineering	1290
Social Science	2140
Agriculture	1010

Problems 15 and 16

The state of Delaware is divided into three counties: Kent, New Castle, and Sussex. The 2002 population totals for each county and the area of each county are given in the following table.

County	2002 Population	Area in Square Miles
Kent	131,069	590
New Castle	512,370	426
Sussex	163,946	938
Total	807,385	1954

Source: U.S. Census Bureau

15. Suppose the 2002 state budget allows for 360 state police officers. How would the officers be apportioned to the three Delaware counties if Jefferson's method is used and is based on population totals?

16. Suppose the 2002 state budget allows for 360 state police officers. How would the officers be apportioned to the three Delaware counties if Jefferson's method is used and is based on county area?

Problems 17 and 18

In 1791, Thomas Jefferson helped to convince President George Washington to veto a bill that established a 120-member House of Representatives to be apportioned using Hamilton's method. When the House could not override Washington's veto, a new bill was passed that established a 105-member House to be apportioned using Jefferson's method. In 1800, there were 141 seats in the House. The following table contains apportionment population totals for the states that were part of the United States in the years 1790 and 1800.

State	1790	1800
Connecticut	236,841	250,622
Delaware	55,540	61,812
Georgia	70,835	138,807
Kentucky	68,705	204,822
Maryland	278,514	306,610
Massachusetts	475,327	574,564
New Hampshire	141,822	183,855
New Jersey	179,570	206,181
New York	331,589	577,805
North Carolina	353,523	424,785
Pennsylvania	432,879	601,863
Rhode Island	68,446	68,970
South Carolina	206,236	287,131
Tennessee	Not a state until 1796	100,169
Vermont	85,533	154,465
Virginia	630,560	747,362
Total	3,615,920	4,889,823

Source: Balinski and Young, *Fair Representation Meeting the Ideal of One Man, One Vote*, 2nd Ed. Washington, DC: Brookings Institution Press, 2001.

17. Use the 1790 apportionment population totals and Jefferson's method to apportion the 105 House seats.

18. Use the 1800 apportionment population totals and Jefferson's method to apportion the 141 House seats.

19. Every apportionment method has drawbacks. Suppose six school board seats will be apportioned to three zones with populations 100, 110, and 150. There is a problem with Jefferson's method in this case. Explain why there is a problem. Consider how the divisor must be modified and how the quotas are affected.

20. Use Webster's method to apportion the six school board seats to three zones with populations 100, 110, and 150. These seats could not be apportioned by Jefferson's method in the previous problem.

Problems 21 and 22

At Apportion High School, class sizes are kept small. The school boasts that their average class size is less than 20. The school's only Spanish teacher can teach 5 sections of Spanish each day. Classes are apportioned based on enrollment.

21. Enrollment figures for the fall quarter are listed in the following table. Apportion the five fall-term sections of Spanish classes according to Jefferson's method and Webster's method. Which apportionment method do you think works best in this situation? Explain.

Course	Number of Students
Beginning Spanish	53
Intermediate Spanish	34
Conversational Spanish	15
Total	102

22. Enrollment figures for the winter quarter are listed in the following table. An advanced Spanish class was added to the schedule, and a part-time instructor was hired to teach two classes in addition to the five that the full-time Spanish teacher covers. Apportion the seven winter-term sections of Spanish classes according to Jefferson's method and Webster's method. Which apportionment method do you think works best in this situation? Explain.

Course	Number of Students
Beginning Spanish	69
Intermediate Spanish	46
Advanced Spanish	12
Conversational Spanish	25
Total	152

23. In problem 15, Jefferson's method was used to assign 360 police officers to three Delaware counties based on 2002 populations. Use Webster's method to apportion the police officers based on population.

24. In problem 16, Jefferson's method was used to assign 360 police officers to three Delaware counties based on the geographical areas of the counties. Use Webster's method to apportion the police officers based on area.

Problems 25 through 28

Under the constitution of the Republic of Freedonia, the legislature must be reapportioned every 10 years based on the census. The most recent census figures for the five states of Freedonia are given in the following table.

State	Population
A	1,592,000
B	1,596,000
C	5,462,000
D	1,323,000
E	1,087,000

25. Apportion the 200-seat legislature using Jefferson's method.

26. Apportion the 200-seat legislature using Webster's method.

27. If Freedonia wanted to preserve the ratio of one legislative seat for approximately every 50,000 citizens, how many seats should the legislature have? Reapportion the legislature using Jefferson's method and the number of seats just determined.

28. Reapportion the legislature using Webster's method and the number of seats determined in problem 27.

Problems 29 through 32

The following table gives the populations in the five counties of Hawaii in 1990 and in 2000.

County	1990 Population	2000 Population
Hawaii	120,317	148,677
Honolulu	836,231	876,156
Kalawao	130	147
Kauai	51,177	58,463
Maui	100,374	128,094
Total	1,108,229	1,211,537

Source: U.S. Census Bureau.

29. Suppose that in 1990 Hawaii decided to create a 50-seat county advisory board. Use Jefferson's method to apportion the seats to the five counties.

30. In the year 2000, the seats on the advisory board needed to be reapportioned because of the changes in the population totals. Reapportion the seats using Jefferson's method to the five counties.

31. Repeat problem 29 using Webster's method. Do Jefferson's method and Webster's method yield different apportionments?

32. Repeat problem 30 using Webster's method. Do Jefferson's method and Webster's method yield different apportionments?

Extended Problems

Problems 33 through 36

While many different apportionment methods have been devised and advocated, only four (Hamilton, Jefferson, Webster, and Huntington–Hill) have been implemented with the U.S. House of Representatives. One method was advocated in 1832 by John Quincy Adams, the sixth president of the United States, and bears his name. In **Adams' method**, also known as the method of smallest divisors, a modified divisor is chosen so that all modified quotas can be rounded upward and have a sum that equals the number of seats to be apportioned; this is in contrast to Jefferson's method, in which all modified quotas are rounded downward.

33. Use Adams' method to apportion the 20 bottles of wine from problem 11.

34. Use Adams' method to apportion the 360 police officers in problem 15.

35. Use Adams' method to apportion the 105 House seats in problem 17.

36. Consider the following statement. "Jefferson's method favors large states, while Adams' method favors small states." Use results from problems 11, 15, 17, 33, 34, and 35 and your knowledge of the rounding techniques for both methods to provide evidence that the statement is true.

Problems 37 through 40

In 1832, a professor at Dartmouth, James Dean, proposed yet another apportionment method called **Dean's method**. Dean's method has never been used to apportion seats for the House of Representatives. Nonetheless, it has been a part of the apportionment debate in the United States. After the 1990 reapportionment, Montana and Massachusetts challenged the constitutionality of the Huntington–Hill method that has been used for years. Faced with losing one of its two House seats, Montana favored Adams' method or Dean's method, either of which would have allowed Montana to retain its two seats but would have left Massachusetts with only 10 seats. Massachusetts suggested using Webster's method, which would have allocated 11 seats to Massachusetts and 1 seat to Montana.

Dean's method is a divisor method and uses the harmonic mean of two numbers. For two whole numbers a and b, the **harmonic mean** is $\frac{2ab}{a+b}$. In Dean's method, also called the harmonic mean method, a modified divisor is chosen so that all modified quotas can be rounded upward or downward, and the resulting whole numbers will add up to the number of seats to be apportioned. If the modified quota is less than the harmonic mean of the nearest whole numbers above and below the modified quota, then round down; otherwise, round up. For example, the modified quota 3.15 is between the whole numbers 3 and 4, and the harmonic mean of 3 and 4 is $\frac{2(3)(4)}{3+4} = \frac{24}{7} \approx 3.43$. Because 3.15 is less than 3.43, we round the modified quota down to 3.

37. Use Dean's method to apportion the 20 bottles of wine in problem 11.

38. Use Dean's method to apportion the 360 police officers in problem 15.

39. Use Dean's method to apportion the 105 House seats in problem 17.

40. Consider the results of problems 37, 38, and 39. Compare the apportionments that result from Dean's method to the apportionments that result from Jefferson's method. It is known that Jefferson's method favors large states. Does Dean's method appear to favor large states, small states, or neither? Support your observation by discussing the way Dean's method rounds modified quotients

41. In 2000, the population of the District of Columbia was 572,059. Research and explain why the population of the District of Columbia is not included with the apportionment population of the United States.

42. The following table contains the 2000 apportionment population totals and standard quotas for each state listed from the most populated to the least populated. The apportionment population of the United States in 2000 was 281,424,177.

a. Apportion the 435 House seats according to Hamilton's method as described in Section 5.1.

b. Apportion the 435 House seats according to Jefferson's method.

c. Compare the apportionments from Hamilton's method and Jefferson's method, paying attention to the states where results of the two methods differ. Which states would prefer that Hamilton's method be used? Which states would prefer that Jefferson's method be used?

d. Based on your comparison of apportionments in part (c), would you say that one (or both) of these two methods appears to favor states with larger populations? Explain.

43. As a result of the reapportionment after the 2000 census, Montana, Wyoming, and Rhode Island each retained the same number of seats, namely 1, 1, and 2, respectively. California gained a seat, for a total of 53 seats. See the table in problem 42 for the year 2000 apportionment population totals. Are these states overrepresented or underrepresented in the House of Representatives? What other states are overrepresented or underrepresented? Explain. (For a full listing of the apportionment following the 2000 census, go to www.census.gov/population/cen2000/tab01.pdf.)

State	2000 Apportionment Population	Standard Quota	State	2000 Apportionment Population	Standard Quota
California	33,930,798	52.447	South Carolina	4,025,061	6.222
Texas	20,903,994	32.312	Oklahoma	3,458,819	5.346
New York	19,004,973	23.376	Oregon	3,428,543	5.300
Florida	16,028,890	24.776	Connecticut	3,409,535	5.270
Illinois	12,439,042	19.227	Iowa	2,931,923	4.532
Pennsylvania	12,300,670	19.013	Mississippi	2,852,927	4.410
Ohio	11,374,540	17.582	Kansas	2,693,824	4.164
Michigan	9,955,829	15.389	Arkansas	2,679,733	4.142
New Jersey	8,424,354	13.022	Utah	2,236,714	3.457
Georgia	8,206,975	12.686	Nevada	2,002,032	3.095
North Carolina	8,067,673	12.470	New Mexico	1,823,821	2.819
Virginia	7,100,702	10.976	West Virginia	1,813,077	2.802
Massachusetts	6,355,568	9.824	Nebraska	1,715,369	2.651
Indiana	6,090,782	9.415	Idaho	1,297,274	2.005
Washington	5,908,684	9.133	Maine	1,277,731	1.975
Tennessee	5,700,037	8.811	New Hampshire	1,238,415	1.914
Missouri	5,606,260	8.666	Hawaii	1,216,642	1.881
Wisconsin	5,371,210	8.302	Rhode Island	1,049,662	1.622
Maryland	5,307,886	8.204	Montana	905,316	1.399
Arizona	5,140,683	7.946	Delaware	785,068	1.213
Minnesota	4,925,670	7.614	South Dakota	756,874	1.170
Louisiana	4,480,271	6.925	North Dakota	643,756	0.995
Alabama	4,461,130	6.896	Alaska	628,933	0.972
Colorado	4,311,882	6.665	Vermont	609,890	0.943
Kentucky	4,049,431	6.259	Wyoming	495,304	0.766

Source: U.S. Census Bureau.

As can be seen from Table 5.45, all four methods of apportionment either violate the quota rule or give rise to one of the paradoxes discussed in this section. Unfortunately, as mathematicians Michel L. Balinski and H. Peyton Young proved (**Balinski and Young's impossibility theorem**), there is *no* apportionment method that satisfies the quota rule and always avoids the Alabama, population, and new-states paradoxes. You may choose or design an apportionment method that obeys the quota rule, but it will be susceptible to at least one of the three paradoxes. You may choose or design an apportionment method that avoids the paradoxes, but then it will violate the quota rule. In other words, there is no perfect apportionment method. The choice of an apportionment method is ultimately a political decision. It is a disappointing realization that the democratic ideal of "one person–one vote" can never be perfectly achieved, although we can come close.

SOLUTION OF THE INITIAL PROBLEM

The school district in which you live receives a grant to buy 25 computers. The superintendent decides to apportion these computers among the schools by using Hamilton's method. The school in your neighborhood is to receive six computers. When the purchase is made, however, a price decrease allows the purchase of 26 computers rather than 25. On hearing this, the neighborhood school's principal reportedly says, "That's good news for the district, but it means our school will get only five computers." What is the principal talking about? Shouldn't every school still get at least the same number of computers, except that one school gets an additional computer?

SOLUTION The principal's claim may sound a little crazy, but he may well be correct. Without more information (knowing what method was used to assign computers to schools, for example), we cannot check the mathematics behind the computer allocations. However, losing a computer in this way would be an example of the Alabama paradox, which can happen under Hamilton's method of apportionment, as we have seen in this section.

PROBLEM SET 5.3

1. A school district assigned 35 instructional assistants to five schools based on enrollment figures. The budget allowed for the hiring of 2 additional assistants. Consider the following apportionment numbers before and after the increase in instructional assistants. Is this an example of a paradox? If so, which paradox has occurred? Explain.

School	Original Apportionment	New Apportionment
Cascades	9	9
Seven Oak	11	10
Riverview	6	7
Pioneer	4	5
Hamilton Creek	5	6

2. A school district assigned 35 instructional assistants to five schools based on enrollment figures. Just before school starts, Riverview and Pioneer report a surge of incoming students. Enrollment at Riverview increased by 9%, while Pioneer had an increase of 7%. As a result, the 35 instructional assistants were reapportioned. Consider the following apportionment numbers before and after the increase in enrollment. Is this an example of a paradox? If so, which paradox has occurred? Explain.

School	Original Apportionment	New Apportionment
Cascades	9	9
Seven Oak	11	11
Riverview	6	5
Pioneer	4	5
Hamilton Creek	5	5

3. Suppose when Oklahoma joined the United States, it received its fair share of congressional seats, but Maryland lost a seat to Virginia. Is this an example of a paradox? If so, which paradox has occurred? Explain.

4. Three survivors of a shipwreck apportion high-calorie food tablets according to the person's size. Survivor I gets 6 tablets, survivor II gets 5 tablets, and survivor III gets 7 tablets. Later, they find another tablet and decide to reapportion the supply. As a result of the reapportionment, Survivor I gets 6 tablets, survivor II gets 6 tablets, and survivor III gets 7 tablets. Is this an example of a paradox? If so, which paradox has occurred? Explain.

5. Hamilton's method, Jefferson's method, or Webster's method was used to apportion representatives to the five states listed in the following table.

State	Modified Quotas	Apportionment
A	23.213	23
B	18.565	19
C	11.993	12
D	15.443	15
E	18.499	18

   a. Which method was used? Explain.
   b. Can you tell whether the quota rule was violated in this apportionment? Explain.

6. Hamilton's method, Jefferson's method, or Webster's method was used to apportion representatives to the five states listed in the following table.

State	Modified Quotas	Apportionment
A	23.213	23
B	18.565	18
C	11.993	11
D	15.443	15
E	18.499	18

   a. Which method was used? Explain.
   b. Can you tell whether the quota rule was violated in this apportionment? Explain.

Problems 7 through 12

A country with three states has 24 seats in the national assembly. The current populations of the states are as follows:

State	Population
Medina	530,000
Alvare	990,000
Loranne	2,240,000

7. a. Apportion the 24 assembly seats using Hamilton's method.
   b. Suppose the national assembly voted to add a seat. Reapportion the 25 assembly seats using Hamilton's method.
   c. Compare the apportionments from parts (a) and (b). Explain what happened to the apportionments. If a paradox arose, identify which one.

8. Suppose the national assembly has 26 seats.
   a. Apportion the 26 assembly seats using Hamilton's method.
   b. A seat is added to the assembly. Apportion the 27 seats using Hamilton's method.
   c. Compare the apportionments from parts (a) and (b). Explain what happened to the apportionments. If a paradox arose, identify which one.

9. a. Apportion the original 24 seats of the national assembly of the country using Jefferson's method.
   b. Does the Alabama paradox arise under Jefferson's method when the number of seats is increased from 24 to 25? Justify your answer.
   c. Does the Alabama paradox arise under Jefferson's method when the number of seats is increased from 25 to 26? Justify your answer.

10. a. Apportion the original 24 seats of the national assembly of the country using Webster's method.
    b. Does the Alabama paradox arise under Webster's method when the number of seats is increased from 24 to 25? Justify your answer.
    c. Does the Alabama paradox arise under Webster's method when the number of seats is increased from 25 to 26? Justify your answer.

11. Suppose that 10 years from now the states have the following populations.

State	Population
Medina	680,000
Alvare	1,250,000
Loranne	2,570,000

a. Use Hamilton's method to apportion 24 national assembly seats.

b. Compare the apportionment from part (a) to the apportionment from problem 7(a). Does the population paradox arise in this case? Explain.

12. For the population totals from the previous problem, use Hamilton's method to apportion 25 seats. Compare the results to the apportionment from problem 7(b). Does the population paradox arise in this case? Explain.

13. A small country with three states has 50 seats in the legislature. The populations of the states are as follows:

State	Population
A	99,000
B	487,000
C	214,000

a. Find and interpret the standard divisor. Apportion the seats using Hamilton's method.

b. Suppose a new state with a population of 116,000 is added to the country, and 7 more seats are added to the legislature. Reapportion the seats using Hamilton's method.

c. Compare the apportionments from parts (a) and (b). Explain what happened. If a paradox occurred, which one?

14. Repeat problem 13 using Webster's method.

15. Repeat problem 13 using Jefferson's method.

16. Repeat problem 13 using Lowndes' method.

Problems 17 through 20

We have seen that an apportionment that apportions to each state either the whole number just below the state's standard quota (called the **lower quota**) or the whole number just above the state's standard quota (called the **upper quota**) is said to satisfy the quota rule.

17. Explain why Hamilton's method will never violate the quota rule.

18. Explain why Lowndes' method will never violate the quota rule.

19. Explain why, if Jefferson's method is used and the quota rule is violated, it must be a violation of the upper quota. A method violates the upper quota if it assigns a whole number greater than the upper quota.

20. Explain why if Adams' method (see Extended Problems 33 through 36 in Section 5.2) is used, any violation of the quota rule must be a violation of the lower quota. A method violates the lower quota if it assigns a whole number less than the lower quota.

Problems 21 through 24

Consider a small country with four provinces. The populations of the four provinces are provided in the table.

Province	Population
North	892,000
South	424,000
West	664,000
East	1,162,000

21. Using Hamilton's method, the apportionments based on legislatures having 314, 315, and 316 seats are provided in the following table.

Province	314 Seats	315 Seats	316 Seats
North	89	89	90
South	43	43	42
West	66	67	67
East	116	116	117

Does the Alabama paradox occur? If it does, which provinces benefit and which provinces lose under each apportionment?

22. Using Lowndes' method, the apportionments based on legislatures having 314, 315, and 316 seats are provided in the following table.

Province	314 Seats	315 Seats	316 Seats
North	89	89	90
South	43	43	43
West	66	67	67
East	116	116	116

Does the Alabama paradox occur? If it does, which provinces benefit and which provinces lose under each apportionment?

23. Using Jefferson's method, the apportionments based on legislatures having 314, 315, and 316 seats are provided in the following table.

Province	314 Seats	315 Seats	316 Seats
North	89	90	90
South	42	42	42
West	66	66	67
East	117	117	117

Does the Alabama paradox occur? If it does, which provinces benefit and which provinces lose under each apportionment?

24. Using Webster's method, the apportionments based on legislatures having 314, 315, and 316 seats are provided in the following table.

Province	314 Seats	315 Seats	316 Seats
North	89	89	89
South	42	43	43
West	67	67	67
East	116	116	117

Does the Alabama paradox occur? If it does, which provinces benefit and which provinces lose under each apportionment?

25. In order to manage the large numbers of stray animals in its cities, a state budgets money to hire 100 animal-control officers. The officers will be apportioned to the cities according to their populations, which are shown in the following table.

City	Population
A	25,250
B	142,500
C	61,500
D	14,750

a. Apportion the officers according to Hamilton's method.

b. If City E, with a population of 49,440, wants to be included, how many new officers should be hired, based on the standard divisor in part (a)?

c. Reapportion the officers to the five cities using Hamilton's method and the new, larger number of animal-control officers determined in part (b).

d. Compare the apportionments from parts (a) and (c). Explain what happened, and state which paradox occurred, if any.

26. Repeat the previous problem using Lowndes' method of apportionment.

Problems 27 through 32

The Republic of Freedonia has experienced growth in every state over a 4-year period. Consider the following population data for Freedonia.

State	Old Population	New Population
A	1,320,000	1,370,000
B	1,515,000	1,565,000
C	4,935,000	5,035,000
D	1,118,000	1,218,000
E	1,112,000	1,212,000

27. Calculate the rate of increase for each state in Freedonia using the formula

$$\text{Percent increase} = \frac{\text{new population} - \text{old population}}{\text{old population}} \times 100\%$$

28. List the states in order from smallest percent increase to largest percent increase.

29. Use Lowndes' method to apportion the 200 seats in the legislature based upon the old population and based upon the new population. Compare the two apportionments. Does the population paradox occur? Explain.

30. Use Hamilton's method to apportion the 200 seats in the legislature based upon the old population and based upon the new population. Compare the two apportionments. Does the population paradox occur? Explain.

31. Use Jefferson's method to apportion the 200 seats in the legislature based upon the old population and based upon the new population. Compare the two apportionments. Does the population paradox occur? Explain.

32. Use Webster's method to apportion the 200 seats in the legislature based upon the old population and based upon the new population. Compare the two apportionments. Does the population paradox occur? Explain.

Problems 33 through 36

Suppose a small country admits a new territory, Northwest, to full provincial status with "equal" representation in the legislature. Northwest is a rapidly growing frontier area with a population of 243,000. Because the country's legislature of 314 seats is based roughly on one seat for each 10,000 people, a law is passed to increase the number of seats by 24, to a total of 338 seats, in order to accommodate the new province.

Province	Population
North	892,000
South	424,000
West	664,000
East	1,162,000
Northwest	243,000

33. Use Hamilton's method to apportion the 338 seats in the new legislature. Compare this apportionment to the one done for 314 seats in problem 21. Does the new-states paradox occur? Explain.

34. Use Jefferson's method to apportion the 338 seats in the new legislature. Compare this apportionment to the one done for 314 seats in problem 23. Does the new-states paradox occur? Explain.

35. Use Webster's method to apportion the 338 seats in the new legislature. Compare this apportionment to the one done for 314 seats in problem 24. Does the new-states paradox occur? Explain.

36. Use Lowndes' method to apportion the 338 seats in the new legislature. Compare this apportionment to the one done for 314 seats in problem 22. Does the new-states paradox occur? Explain.

37. Police patrols are apportioned to areas according to crime statistics. Area I reports 40 crimes in a week, Area II reports 82, and Area III reports 285.

a. Apportion 96 police patrols using Hamilton's method.

b. The budget allows for an additional patrol. Apportion 97 police patrols using Hamilton's method.

c. Compare the apportionments from parts (a) and (b). Explain what happened when the number of police patrols was increased. State which paradox occurred, if any.

38. Consider the crime data from the previous problem.

a. Apportion 96 police patrols using Jefferson's method.

b. Apportion 97 police patrols using Jefferson's method.

c. Compare the apportionments from parts (a) and (b). Did anything unusual happen when the number of police patrols was increased? State which paradox occurred, if any.

d. Did either apportionment from parts (a) or (b) violate the quota rule? Explain.

39. Create an example of a country with four states and a population of 5 million in which the Alabama paradox occurs when the number of seats in the legislature is increased from 100 to 102. Use Hamilton's method.

40. Create an example of a country with three states and a population of 5 million in which the Alabama paradox occurs when the number of seats in the legislature is increased from 100 to 101. Use Hamilton's method.

41. Explain why Jefferson's method cannot produce the Alabama paradox.

42. Explain why Jefferson's method cannot produce the new-states paradox.

43. Explain why Webster's method cannot produce the new-states paradox.

44. Explain why Webster's method cannot produce the Alabama paradox.

Extended Problems

Problems 45 through 50: Geometric Mean and Huntington–Hill Method

The method currently being used to apportion the U.S. House of Representatives is the Huntington–Hill method, which has been the official apportionment method since 1941. It is a variation of Webster's method but differs from it in that the decision to round a modified quota is based on whether the modified quota is less than or greater than the geometric mean of the two whole numbers immediately before and after it. The geometric mean of two numbers differs from the arithmetic mean, or "average" of the two numbers. The **geometric mean** of two whole numbers, a and b, is $\sqrt{ab}$. Under the Huntington–Hill method, if the modified quota is greater than the geometric mean of the whole numbers just above and below it, the modified quota is rounded up to obtain the number of seats. If the modified quota is less than the geometric mean of those two numbers, it is rounded down.

For example, in applying the Huntington–Hill method, suppose that the modified quota under consideration is 4.475. The whole number less than 4.475 is 4, and the whole number greater than 4.475 is 5. The geometric mean of $a = 4$ and $b = 5$ is $\sqrt{ab} = \sqrt{20} \approx 4.4721$. Because 4.475 is greater than 4.4721, the modified quota is rounded up to 5. If the modified quota under consideration had been 5.475, the geometric mean of 5 and 6 would have been $\sqrt{5 \times 6} \approx 5.4772$. In this case, the modified quota would have been rounded down to 5, since $5.475 < 5.4772$.

45. a. The fractional part of the geometric mean of two consecutive whole numbers will always be less than 0.5. Check this for five pairs of consecutive numbers.

b. The fractional part of the geometric mean of two consecutive whole numbers will always be greater than 0.41. Check this for five pairs of consecutive numbers.

46. Use the Huntington–Hill method to find the apportionment for each state in a small country with four states. The quotas for each state are as follows:

State	Modified Quota
A	10.47
B	3.47
C	5.47
D	7.59

47. Use the Huntington–Hill method to find the apportionment for each state in a country with 3 states and 40 seats in the legislature. The populations are as follows:

State	Population
A	581,500
B	846,711
C	1,022,600

48. Use the Huntington–Hill method to apportion the 435 U.S. House of Representative seats according to the year 2000 census data from the Extended Problems in section 5.2.

49. The first apportionment of the U.S. House of Representatives was calculated using Jefferson's method. Use the information that follows to determine if the apportionment would have changed if the Huntington–Hill method had been used instead. The 1790 apportionment population totals are given in problem 17 of section 5.2. Use the Huntington–Hill method to reapportion the 105 house seats in 1790. Compare your results with those obtained under Jefferson's method in problem 17 of section 5.2 and those obtained under Hamilton's method in problem 15 of section 5.1. How are the apportionments different?

50. A paradox of the Huntington–Hill method is that two states can have quotas that differ by more than 1 yet have the same apportionment. Create an apportionment example that illustrates this phenomenon.

51. There is currently a movement to discontinue the use of the Huntington–Hill method and revert to Webster's method. Investigate what arguments have been presented to support Webster's method over the Huntington–Hill method. According to H. Peyton Young, the Huntington–Hill method has a "fundamental flaw." What is that flaw? Research the controversy surrounding these two apportionment methods and write a report that summarizes your findings. For information on the Internet, search keywords "Balinski and Young."

52. When and how was the population paradox discovered? Research the population paradox and summarize your findings in a report.