6.1 Evaluate *n*th Roots and Use Rational Exponents

Before You evaluated square roots and used properties of exponents.

Now You will evaluate *n*th roots and study rational exponents.

Why? So you can find the radius of a spherical object, as in Ex. 60.

Key Vocabulary
- ***n*th root of *a***
- **index of a radical**

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$. In general, for an integer n greater than 1, if $b^n = a$, then b is an ***n*th root of *a***. An nth root of a is written as $\sqrt[n]{a}$ where n is the **index** of the radical.

You can also write an nth root of a as a power of a. If you assume the power of a power property applies to rational exponents, then the following is true:

$$(a^{1/2})^2 = a^{(1/2) \cdot 2} = a^1 = a$$

$$(a^{1/3})^3 = a^{(1/3) \cdot 3} = a^1 = a$$

$$(a^{1/4})^4 = a^{(1/4) \cdot 4} = a^1 = a$$

Because $a^{1/2}$ is a number whose square is a, you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer n greater than 1.

KEY CONCEPT *For Your Notebook*

Real *n*th Roots of *a*

Let n be an integer ($n > 1$) and let a be a real number.

	***n* is an even integer.**		***n* is an odd integer.**
$a < 0$	No real nth roots.	$a < 0$	One real nth root: $\sqrt[n]{a} = a^{1/n}$
$a = 0$	One real nth root: $\sqrt[n]{0} = 0$	$a = 0$	One real nth root: $\sqrt[n]{0} = 0$
$a > 0$	Two real nth roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$	$a > 0$	One real nth root: $\sqrt[n]{a} = a^{1/n}$

EXAMPLE 1 Find *n*th roots

Find the indicated real *n*th root(s) of *a*.

a. $n = 3, a = -216$

b. $n = 4, a = 81$

Solution

a. Because $n = 3$ is odd and $a = -216 < 0$, -216 has one real cube root. Because $(-6)^3 = -216$, you can write $\sqrt[3]{-216} = -6$ or $(-216)^{1/3} = -6$.

b. Because $n = 4$ is even and $a = 81 > 0$, 81 has two real fourth roots. Because $3^4 = 81$ and $(-3)^4 = 81$, you can write $\pm\sqrt[4]{81} = \pm 3$ or $\pm 81^{1/4} = \pm 3$.

RATIONAL EXPONENTS A rational exponent does not have to be of the form $\frac{1}{n}$. Other rational numbers such as $\frac{3}{2}$ and $-\frac{1}{2}$ can also be used as exponents. Two properties of rational exponents are shown below.

KEY CONCEPT *For Your Notebook*

Rational Exponents

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

$$a^{m/n} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m$$

$$a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{\left(a^{1/n}\right)^m} = \frac{1}{\left(\sqrt[n]{a}\right)^m},\ a \neq 0$$

EXAMPLE 2 Evaluate expressions with rational exponents

Evaluate (a) $16^{3/2}$ and (b) $32^{-3/5}$.

Solution

	Rational Exponent Form	Radical Form
a.	$16^{3/2} = \left(16^{1/2}\right)^3 = 4^3 = 64$	$16^{3/2} = \left(\sqrt{16}\right)^3 = 4^3 = 64$
b.	$32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{\left(32^{1/5}\right)^3} = \frac{1}{2^3} = \frac{1}{8}$	$32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{\left(\sqrt[5]{32}\right)^3} = \frac{1}{2^3} = \frac{1}{8}$

AVOID ERRORS
Be sure to use parentheses to enclose a rational exponent: 9^(1/5) ≈ 1.552. Without them, the calculator evaluates a power and then divides: 9^1/5 = 1.8.

EXAMPLE 3 Approximate roots with a calculator

	Expression	Keystrokes	Display
a.	$9^{1/5}$	9 [^] [(] 1 [÷] 5 [)] [ENTER]	1.551845574
b.	$12^{3/8}$	12 [^] [(] 3 [÷] 8 [)] [ENTER]	2.539176951
c.	$\left(\sqrt[4]{7}\right)^3 = 7^{3/4}$	7 [^] [(] 3 [÷] 4 [)] [ENTER]	4.303517071

✓ **GUIDED PRACTICE** for Examples 1, 2, and 3

Find the indicated real nth root(s) of a.

1. $n = 4, a = 625$

2. $n = 6, a = 64$

3. $n = 3, a = -64$

4. $n = 5, a = 243$

Evaluate the expression without using a calculator.

5. $4^{5/2}$

6. $9^{-1/2}$

7. $81^{3/4}$

8. $1^{7/8}$

Evaluate the expression using a calculator. Round the result to two decimal places when appropriate.

9. $4^{2/5}$

10. $64^{-2/3}$

11. $\left(\sqrt[4]{16}\right)^5$

12. $\left(\sqrt[3]{-30}\right)^2$

EXAMPLE 4 Solve equations using *n*th roots

Solve the equation.

a. $4x^5 = 128$

$x^5 = 32$ — **Divide each side by 4.**

$x = \sqrt[5]{32}$ — **Take fifth root of each side.**

$x = 2$ — **Simplify.**

b. $(x - 3)^4 = 21$

$x - 3 = \pm\sqrt[4]{21}$ — **Take fourth roots of each side.**

$x = \pm\sqrt[4]{21} + 3$ — **Add 3 to each side.**

$x = \sqrt[4]{21} + 3$ or $x = -\sqrt[4]{21} + 3$ — **Write solutions separately.**

$x \approx 5.14$ or $x \approx 0.86$ — **Use a calculator.**

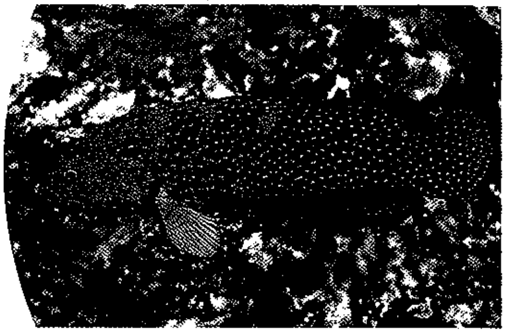
AVOID ERRORS
When n is even and $a > 0$, be sure to consider both the positive and negative nth roots of a.

EXAMPLE 5 Use *n*th roots in problem solving

BIOLOGY A study determined that the weight w (in grams) of coral cod near Palawan Island, Philippines, can be approximated using the model

$$w = 0.0167\ell^3$$

where ℓ is the coral cod's length (in centimeters). Estimate the length of a coral cod that weighs 200 grams.



Solution

$w = 0.0167\ell^3$ — **Write model for weight.**

$\mathbf{200} = 0.0167\ell^3$ — **Substitute 200 for *w*.**

$11{,}976 \approx \ell^3$ — **Divide each side by 0.0167.**

$\sqrt[3]{11{,}976} \approx \ell$ — **Take cube root of each side.**

$22.9 \approx \ell$ — **Use a calculator.**

▸ A coral cod that weighs 200 grams is about 23 centimeters long.

✓ GUIDED PRACTICE for Examples 4 and 5

Solve the equation. Round the result to two decimal places when appropriate.

13. $x^3 = 64$ **14.** $\frac{1}{2}x^5 = 512$ **15.** $3x^2 = 108$

16. $\frac{1}{4}x^3 = 2$ **17.** $(x - 2)^3 = -14$ **18.** $(x + 5)^4 = 16$

19. WHAT IF? Use the information from Example 5 to estimate the length of a coral cod that has the given weight.

a. 275 grams **b.** 340 grams **c.** 450 grams

6.1 EXERCISES

HOMEWORK KEY ○ = **WORKED-OUT SOLUTIONS** on p. WS12 for Exs. 9, 25, and 63
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 33, 46, 47, and 65

SKILL PRACTICE

1. **VOCABULARY** Copy and complete: In the expression $\sqrt[4]{10{,}000}$, the number 4 is called the __?__.

2. ★ **WRITING** *Explain* how the sign of a determines the number of real fourth roots of a and the number of real fifth roots of a.

EXAMPLE 1 on p. 414 for Exs. 3–20

MATCHING EXPRESSIONS **Match the expression in rational exponent notation with the equivalent expression in radical notation.**

3. $2^{1/3}$ 4. $2^{3/2}$ 5. $2^{2/3}$ 6. $2^{1/2}$

A. $(\sqrt{2})^3$ B. $\sqrt{2}$ C. $\sqrt[3]{2}$ D. $(\sqrt[3]{2})^2$

USING RATIONAL EXPONENT NOTATION **Rewrite the expression using rational exponent notation.**

7. $\sqrt[3]{12}$ 8. $\sqrt[5]{8}$ (9.) $(\sqrt[3]{10})^7$ 10. $(\sqrt[8]{15})^3$

USING RADICAL NOTATION **Rewrite the expression using radical notation.**

11. $5^{1/4}$ 12. $7^{1/3}$ 13. $14^{2/5}$ 14. $21^{9/4}$

FINDING *N*TH ROOTS **Find the indicated real nth root(s) of a.**

15. $n = 2, a = 64$ 16. $n = 3, a = -27$ 17. $n = 4, a = 0$

18. $n = 3, a = 343$ 19. $n = 4, a = -16$ 20. $n = 5, a = -32$

EXAMPLE 2 on p. 415 for Exs. 21–33

EVALUATING EXPRESSIONS **Evaluate the expression without using a calculator.**

21. $\sqrt[6]{64}$ 22. $8^{1/3}$ 23. $16^{3/2}$ 24. $\sqrt[3]{-125}$

(25.) $27^{2/3}$ 26. $(-243)^{1/5}$ 27. $(\sqrt[3]{8})^{-2}$ 28. $(\sqrt[3]{-64})^4$

29. $(\sqrt[4]{16})^{-7}$ 30. $25^{3/2}$ 31. $64^{-2/3}$ 32. $\dfrac{1}{81^{-3/4}}$

33. ★ **MULTIPLE CHOICE** What is the value of $128^{5/7}$?

(A) 8 (B) 16 (C) 32 (D) 64

EXAMPLE 3 on p. 415 for Exs. 34–46

APPROXIMATING ROOTS **Evaluate the expression using a calculator. Round the result to two decimals places when appropriate.**

34. $\sqrt[5]{32{,}768}$ 35. $\sqrt[7]{1695}$ 36. $\sqrt[9]{-230}$ 37. $85^{1/6}$

38. $25^{-1/3}$ 39. $20{,}736^{1/4}$ 40. $(\sqrt[4]{187})^3$ 41. $(\sqrt{6})^{-5}$

42. $(\sqrt[5]{-8})^8$ 43. $86^{-5/6}$ 44. $1974^{2/7}$ 45. $\dfrac{1}{(-17)^{3/5}}$

46. ★ **MULTIPLE CHOICE** Which expression has the greatest value?

(A) $27^{3/5}$ (B) $5^{3/2}$ (C) $\sqrt[3]{81}$ (D) $(\sqrt[3]{2})^8$

47. ★ **OPEN-ENDED MATH** Write two different expressions of the form $a^{1/n}$ that equal 3, where a is a real number and n is an integer greater than 1.

riate.

6

EXAMPLE 4
on p. 416
for Exs. 48–58

ERROR ANALYSIS *Describe* and correct the error in solving the equation.

48.
$x^3 = 27$
$x = \sqrt[3]{27}$
$x = 9$
✗

49.
$x^4 = 81$
$x = \sqrt[4]{81}$
$x = 3$
✗

SOLVING EQUATIONS Solve the equation. Round the result to two decimal places when appropriate.

50. $x^3 = 125$ **51.** $5x^3 = 1080$ **52.** $x^6 + 36 = 100$

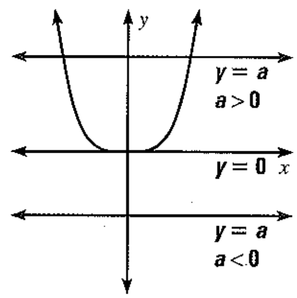
53. $(x - 5)^4 = 256$ **54.** $x^5 = -48$ **55.** $7x^4 = 56$

56. $x^3 + 40 = 25$ **57.** $(x + 10)^5 = 70$ **58.** $x^6 - 34 = 181$

59. CHALLENGE The general shape of the graph of $y = x^n$, where n is a positive *even* integer, is shown in red.

a. *Explain* how the graph justifies the results in the Key Concept box on page 414 when n is a positive *even* integer.

b. Draw a similar graph that justifies the results in the Key Concept box when n is a positive *odd* integer.



PROBLEM SOLVING

EXAMPLE 5
on p. 416
for Exs. 60–65

60. SHOT PUT The shot used in men's shot put has a volume of about 905 cubic centimeters. Find the radius of the shot. (*Hint:* Use the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere.)

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61. BOWLING A bowling ball has a surface area of about 232 square inches. Find the radius of the bowling ball. (*Hint:* Use the formula $S = 4\pi r^2$ for the surface area of a sphere.)

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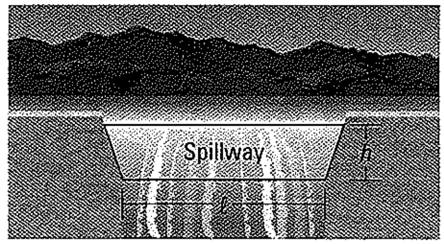
62. INFLATION If the average price of an item increases from p_1 to p_2 over a period of n years, the annual rate of inflation r (expressed as a decimal) is given by $r = \left(\frac{p_2}{p_1}\right)^{1/n} - 1$. Find the rate of inflation for each item in the table. Write each answer as a percent rounded to the nearest tenth.

Item	Price in 1950	Price in 1990
Butter (lb)	\$.7420	\$2.195
Chicken (lb)	\$.4430	\$1.087
Eggs (dozen)	\$.6710	\$1.356
Sugar (lb)	\$.0936	\$.4560

63. MULTI-STEP PROBLEM The power p (in horsepower) used by a fan with rotational speed s (in revolutions per minute) can be modeled by the formula $p = ks^3$ for some constant k. A certain fan uses 1.2 horsepower when its speed is 1700 revolutions per minute. First find the value of k for this fan. Then find the speed of the fan if it uses 1.5 horsepower.

○ = **WORKED-OUT SOLUTIONS on p. WS1** ★ = **STANDARDIZED TEST PRACTICE**

64. **WATER RATE** A *weir* is a dam that is built across a river to regulate the flow of water. The flow rate Q (in cubic feet per second) can be calculated using the formula $Q = 3.367\ell h^{3/2}$ where ℓ is the length (in feet) of the bottom of the spillway and h is the depth (in feet) of the water on the spillway. Determine the flow rate of a weir with a spillway that is 20 feet long and has a water depth of 5 feet.



65. ★ **EXTENDED RESPONSE** Some games use dice in the shape of regular polyhedra. You are designing dice and want them all to have the same volume as a cube with an edge length of 16 millimeters.

Name	Tetrahedron	Octahedron	Dodecahedron	Icosahedron
				
Number of faces	4	8	12	20
Volume formula	$V = 0.118x^3$	$V = 0.471x^3$	$V = 7.663x^3$	$V = 2.182x^3$

a. Find the volume of a cube with an edge length of 16 millimeters.

b. Find the edge length x for each of the polyhedra shown in the table.

c. Does the polyhedron with the greatest number of faces have the smallest edge length? *Explain.*

66. **CHALLENGE** The mass of the particles that a river can transport is proportional to the sixth power of the speed of the river. A certain river normally flows at a speed of 1 meter per second. What must its speed be in order to transport particles that are twice as massive as usual? 10 times as massive? 100 times as massive?

MIXED REVIEW

Evaluate the expressions for the given values of x and y. ***(p. 10)***

67. $\dfrac{x+3y}{x-y}$ when $x = 3$ and $y = 5$

68. $\dfrac{4x-y}{x-2y}$ when $x = 6$ and $y = -2$

Find all the zeros of the function.

69. $f(x) = x^2 - 2x - 35$ ***(p. 252)***

70. $f(x) = x^2 - 8x + 25$ ***(p. 292)***

71. $f(x) = x^3 - 8x^2 + 4x - 32$ ***(p. 379)***

72. $f(x) = x^3 + 4x^2 + 25x + 100$ ***(p. 379)***

73. $f(x) = x^4 - 3x^3 - 31x^2 + 63x + 90$ ***(p. 379)***

74. $f(x) = x^4 + 10x^3 + 25x^2 - 36$ ***(p. 379)***

PREVIEW Prepare for Lesson 6.2 in Exs. 75–82.

Simplify the expression. Tell which properties of exponents you used. ***(p. 330)***

75. $\dfrac{x^{-4}}{x^3}$

76. $(x^4)^{-3}$

77. $(3x^2y)^{-3}$

78. $4x^0y^{-4}$

79. $x^6 \cdot x^{-2}$

80. $\left(\dfrac{x^3}{y^{-2}}\right)^2$

81. $\dfrac{4x^3y^6}{10x^5y^{-3}}$

82. $\dfrac{3x}{x^3y^2} \cdot \dfrac{y^4}{9x^{-2}}$